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# NEUTRON INTERACTION IN FISSILE ASSEMBLIES

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**Abstract**—Existing methods for assessing the degree of criticality of an array of interacting fissile units are reviewed from a practical viewpoint. Emphasis is placed upon those methods which rely mainly upon direct experimental evidence and special reference is made to the "interaction parameter" method.

## 1. INTRODUCTION

WHEN two or more pieces of fissile material are brought near together, the neutron output of each piece exceeds its output in isolation due to neutron exchange and multiplication between the pieces. This interaction between fissile bodies in array can lead to the system becoming critical even when each individual unit is well subcritical, and an assessment of its effect is of great importance in the safe and economic working of plants which produce and fabricate fissile material and in the associated problems of transport and storage. One solution to the interaction problem is to place the separate units far enough apart for it to be obvious that interaction is negligible. In most practical cases this solution is grossly uneconomic, and there is evidently a need for techniques which will provide estimates of the minimum safe spacing (or maximum number of objects) which are as realistic as possible and demonstrably conservative.

The present work describes the methods which have been developed at three major establishments concerned with interaction problems for assessing the degree of criticality of arrays of fissile bodies. The description is limited to those methods which rely primarily upon experimental data, whether it be obtained from direct interaction experiments or from multiplication measurements on allied systems. Purely theoretical approaches to the problem have been made[1–4] but in the main they are restricted to applications of two or multigroup diffusion theory with the result that in certain cases the intrinsic errors of the method are unpredictable and may lead to non-conservative results. It is for this reason that the semi-experimental methods described below are believed to be more consistent and therefore more satisfactory from the safety point of view.

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The published work of Paxton *et al.* of the Los Alamos Scientific Laboratory deals almost exclusively with arrays of $U^{235}$ metal spheres, and whilst it is the most direct method of the three to be described it is also the most limited in application. Cubic and plane arrays of the spheres are constructed and external multiplication measurements made. The number of bodies required to give criticality in the environments studied is then estimated by extrapolation. A theoretical correlation of the data has been attempted by Ketzlach of Hanford, although no great detail is available.

The method developed by Henry and co-workers at the Oak Ridge Gaseous Diffusion Plant in conjunction with Callihan *et al.* of the Oak Ridge National Laboratory is designed primarily to deal with fissile solutions and correlates theoretical two-group multiplication factors ($k$) with measured solid angles in critical arrays. In this way a recipe for the safe arrangement of solution containers is obtained.

The "interaction parameter" method used at the Atomic Weapons Research Establishment, Aldermaston, derives in a very general way conditions for the safety of arrays of fissile bodies in terms of the neutron outputs induced in the bodies by unit sources placed at the positions of the other bodies. Such outputs can either be measured experimentally or estimated theoretically. This method is given most space in the present account for the reason that it has not previously been written up in detail in open literature, although certain mathematical aspects have been published elsewhere by Mayne[15].

## 2. LOS ALAMOS INTERACTION EXPERIMENTS

Interaction work published openly by Los Alamos consists of a series of lattice experiments carried out by Paxton *et al.* using $U^{235}$ metal spheres[5], [6]. The initial purpose of these experiments was to clear, criticality-wise, storage arrays of particular interest to Los Alamos, and it is only recently that the results have been used by Ketzlach[7] of Hanford as the basis for formulating a systematic method of clearing arrays of spheres in vaults or other communal reflecting material. Unfortunately, Ketzlach's paper is only a preliminary report and contains no great detail. Thus, apart from a brief outline of his proposed method, this section merely presents the experimental information and direct deductions from it. To date, it is the only published data on many-body arrays of high-enrichment metal spheres (essentially fast neutron systems) and as such has been kept apart from the solution work (thermal neutron systems) of Henry and Callihan to be described in Section 3.

### 2.1 *The Measured Multiplications*

In each of the lattices considered by Paxton (*loc. cit.*) the following multiplication factors were measured.

(i) The "self" multiplication factor, defined as the ratio of the external neutron flux of one isolated unit, with a central mock fission source, to that of a natural uranium replica with the same source.

(ii) the "overall" multiplication factor, defined as the ratio of the external neutron flux with a mock fission source placed in the central unit of the array (i.e. with the other units present) to that with the source placed in a natural uranium replica of one unit in isolation.

The "cross" multiplication factor, defined as the ratio of the overall to the self multiplication factor, was deduced from these measurements and gave a measure of the effect of interaction upon the central body of the array.

The reciprocal of overall and cross multiplication was found to vary approximately linearly with the number of units in the lattice (as had been predicted theoretically) and linear extrapolation of these results to zero reciprocal multiplication was used to estimate the number of units required to bring about criticality. A log–log plot of this number as a function of the mean lattice density (units/ft$^3$) produced a linear variation.

### 2.2 *The Lattices Considered and the Results Obtained*

An account of two series of experiments with metal spheres each equivalent in reactivity to 20 kg "oralloy" (93·5% $U^{235}$) is given in reference (5) together with details of the multiplication measurements. The first series was conducted with up to 27 units arranged on a cubic lattice, bare and placed inside 5 ft, 4 ft, and 3 ft concrete vaults with walls 1 ft thick. The second series used up to 5 units on a cubic lattice, bare and inside 3 ft, 2 ft, and 1 ft vaults, again with walls 1 ft thick. In all cases where a vault was used the vault size was adjusted to be three times the lattice spacing.

In both series of experiments the effect of the following factors was considered.

(a) Varying the lattice spacing and number of units.
(b) The presence of vault walls and top.
(c) Tamping of each unit by a contiguous non-uniform reflector of natural uranium; this had the effect of increasing the self-multiplication of each unit from 3 to about 5.
(d) Splitting each unit (except the central one) into two hemispheres laid side by side at the lattice point.
(e) The presence of a paraffin "man" and boron plastic within the vault.

Table 1 (p. 256) gives a summary of the results obtained by extrapolation for the numbers of units in critical cubic arrays.

Only in the case of the 3 ft vault did the reciprocal overall multiplication fall below 0·05; in most cases the lowest value reached was in the range 0·10–0·15. The extrapolations to critical given in Table 1 may therefore be subject to errors of order 10% or more, and so some caution must be used when applying these results to lattices containing more than about three-quarters of the critical number of units shown.

Two points can be deduced from the values listed in Table 1.

(i) The number of critical units minus one is inversely proportional to the

self-multiplication of one unit. That this should be approximately true is apparent from equations (4.7) and (4.16) of Section 4 and is borne out by comparing the results for bare and tamped units in Table 1, the self-multiplications of which are 3 and 5, respectively.

TABLE 1

*Number of* 20 kg Oy* *Units in Critical Cubic Arrays*

| Vault size (ft) | Centre-to-centre spacing (in.) | Lattice density (units/ft$^3$) | Number of units in critical array | | |
|---|---|---|---|---|---|
| | | | Bare | Tamped | Split |
| 5 | 20 | 0·217 | 99 | 60 | — |
| 4 | 16 | 0·422 | 61 | 37 | 73 |
| 3 | 11 | 1·30 | 30 | 19 | 45 |
| 2 | 8 | 3·38 | 12 | — | 16 |
| 1 | Contact | 25·7 | 3·6 | — | — |

*93·5% $U^{235}$.

(ii) If the average density of fissile material in a given size of lattice is maintained but is represented by a larger number of units of smaller mass then the overall degree of criticality of the lattice is reduced, i.e. more "split" units have to be introduced to maintain criticality.

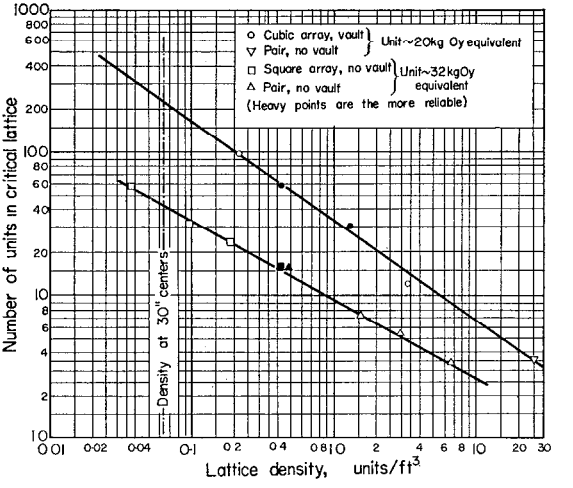


FIG. 1. Approximate critical arrays of similar oralloy metal units.

The results for bare units listed in Table 1 are given graphically in Fig. 1, together with some results listed in reference (6) which refer to experiments done on plane square arrays of units each equivalent to 32 kg oralloy. In each case the plane array was situated 10 in. above a concrete floor.

The effect of the concrete vault walls and top was to increase the overall multiplication of the cubic lattices by factors varying from about 1·2–1·5 for a small number of units to about 2–2·5 for a large number of units. The presence of a paraffin "man" in the vault reduced the overall multiplication by a small amount. The introduction of one type of boron plastic (4·8% B, 43% H, 31% C, 21% O by atoms) increased the overall multiplication, reflection overcoming the absorption in the boron, whilst another type containing more boron (81·3% B, 7·4% H, 11·0% C, 0·7% O and 0·2% Ca) decreased the multiplication.

### 2.3 *Proximity Tests*

As a guide to assessing the effect of interaction in flooded arrays reference (5) gives the variation of reciprocal multiplication as a function of separation distance for (a) two 20 kg Oy units in air, (b) two similar units completely surrounded by water, and (c) 2, 3 and 4 cylinders of oralloy, 4 in. diameter and weighing 13·2 kgm, each contained in a magnesia crucible $4\frac{3}{4}$ in. outer diameter and 11 in. high, surrounded by graphite, $5\frac{3}{8}$ in. outer diameter and 11 in. high, and the whole array completely immersed in water. The results are given in Figs. 2 and 3.

### 2.4 *Correlation of the Lattice Data*

An attempt to correlate the results given in Table 1 into a systematic method for estimating the critical number of units of arbitrary mass and spacing in a cubic arrangement has been carried out recently by Ketzlach (*loc. cit.*). He assumes that the cubic lattices can be replaced by spheres of uniform low density oralloy in infinite reflecting material, and relates the critical radius of such a sphere to the critical radius when the density of the reflector is correspondingly reduced, the latter radius being simply equal to the critical radius for full density core and reflector divided by the factor of reduction. The difference between these two radii, analogous to a reflector savings, he fits empirically to Paxton's results, and indicates very briefly how the modification to arbitrary mass and spacing of units might be carried out.

Paxton's results on "split" units indicate that an homogenized model of a critical array of metal spheres will in fact be subcritical, so that in transforming from Ketzlach's model back to a lumped system the degree of criticality is likely to be increased. This could be an objection to the method, although this difficulty has to some extent been avoided by empirically fitting to experiment. At this stage, however, it would seem unwise to apply the method to units in array which are larger or have higher self-multiplications than the units used by Paxton in his experiments.

Larrick[8] has also used an homogenization technique in treating the storage of MTR-type fuel elements. In the $U^{235}/H_2O$ systems that he considers the dangers brought about by redistributing the fissile material are even more apparent.

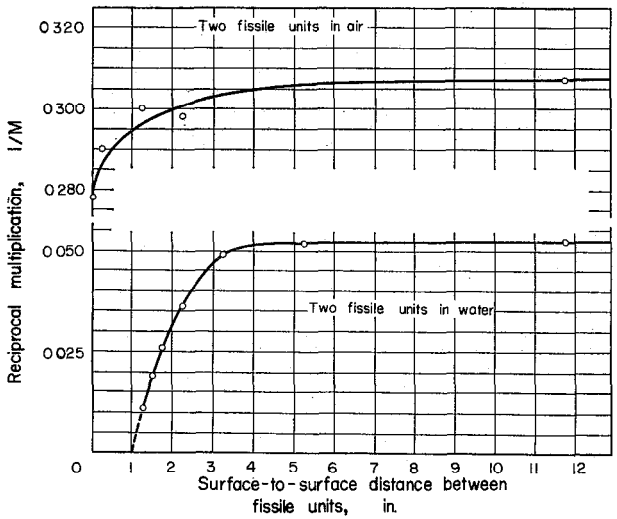


FIG. 2. Proximity test on a pair of fissile units, dry and flooded.

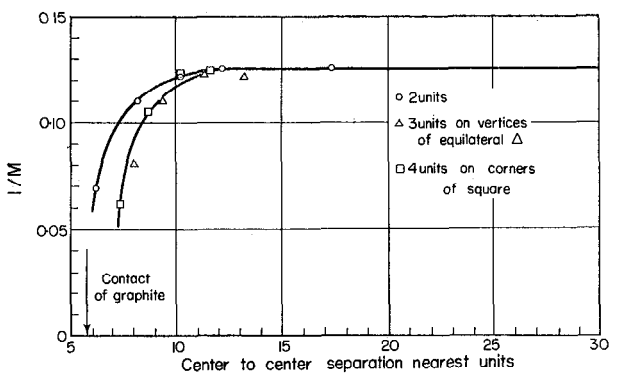


FIG. 3. Proximity test on 2 to 4 oralloy cylinders in melting geometry, flooded. Dry the multiplication of 4 units in contact (5·15 in. centre-to-centre) is less than 3.

## 3. INTERACTION STUDIES AT THE OAK RIDGE GASEOUS DIFFUSION PLANT

Over the past few years interaction studies have been carried out at the Oak Ridge Gaseous Diffusion Plant (ORGDP) with a view to obtaining rules of thumb for designing plant and storing cylinders which are to contain solutions of fissile material, mainly in the form of aqueous solutions of uranyl fluoride ($UO_2F_2$).

For air-spaced arrays of such systems three working principles have been evolved.

(i) Each individual container must be safe (as opposed to being merely subcritical[9]) when completely surrounded by water.

(ii) The container separation may never be less than 12 in.

(iii) Containers must always be separated so that the maximum total solid angle subtended at the most central unit does not exceed some prescribed safe value.

Criteria (i) and (ii) are designed to prevent criticality being caused by accidental or purposeful flooding of the array, since 12 in. of water effectively isolates any one component from its neighbours in the array.

Criterion (iii) refers more specifically to interaction in the array and has been considered in some detail by Henry, Knight and Newlon in a series of ORGDP reports[3], [9], [10]. These authors use the experimental results of Callihan *et al.*[11], [12], [13] on critical assemblies of similar interacting containers. They plot the total solid angle subtended at the most central unit in the critical configuration against the value of $k^*$ for each container in isolation. These $k$-values are obtained by group diffusion methods, as will be indicated later.

### 3.1 *Experimental Basis of the Method*

Knight[10] describes a typical experiment on two interacting cylinders in the following terms.

Two similar cylindrical containers of "carefully measured" dimensions are placed at a known distance apart and a solution containing enriched uranium is pumped in through the bottom of each cylinder by means of pipes. The solution height is kept the same in both cylinders and the level gradually raised until criticality is reached. The geometry at this stage, and the composition of the solution allow the critical solid angle, $\Omega$, and the multiplication factor, $k$, for each container in isolation to be evaluated.

In these experiments the aspect ratios of the cylinders and the $H/U^{235}$ ratio of the solution were varied; for the latter the range 44·3 to 337 was covered at a $U^{235}$ enrichment of 90%. Multibody arrays of bare cylinders and slabs were also examined, and a summary of results for the total solid angles at critical as functions of the $k$-values for individual containers is given in Fig. 4, this being a reproduction of Fig. 1 of reference (9).

*The factor $k$ used here is the multiplication factor per generation, i.e. $k_{eff}$ in reactor theory.

### 3.2 *Safety Criteria for Bare Containers*

From Fig. 4 Henry, Knight, and Newlon conclude that, for *bare* containers:

(a) for units with the same *k-value* but different shapes, interaction increases with increasing solid angle;

(b) the assumption that the total solid angle subtended at the central unit be used for multi-body air-spaced arrays is conservative.

Included in Fig. 4 is the "*calculated* minimum critical curve". This refers to pairs of infinite cylinders* containing the most reactive solution found in the plant ($H/U^{235}$ ratio equal to 44·3 at 90% assay) as calculated from the two

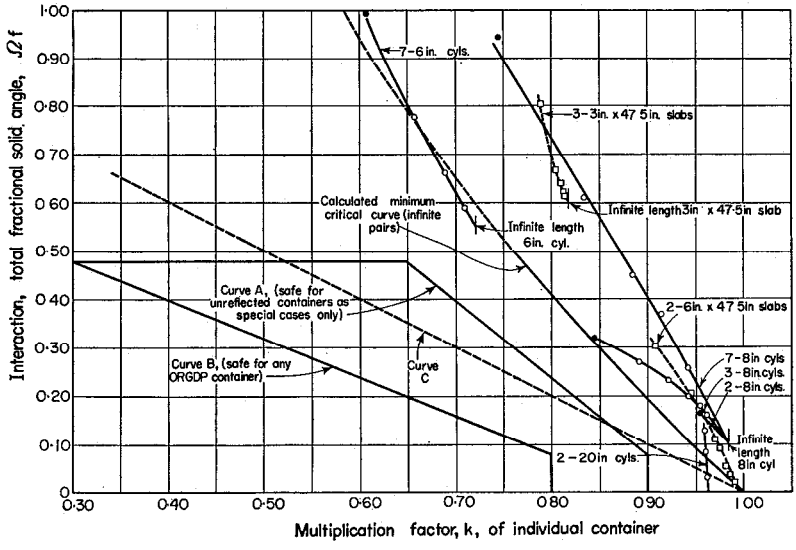


FIG. 4. Interaction vs multiplication factor:
~90% $U^{235}$ assay.
$H/U^{235}$ atomic ratios:
~44·3 for 6 in. and 8 in. cylinders.
~169 for 20 in. cylinders.
~330 for 3 in. and 6 in. slabs.
Variable extrapolation length, *e*, used for slabs.
● Indicates cylinders in contact.

group formulae to be given in Section 3.6. Although the maximum solid angle for two cylinders in contact is about 20% of $4\pi$ the part of the curve for solid angles larger than this value will give a conservative theoretical limit for multi-body systems.

Curve A of Fig. 4 is termed by Newlon[10] "an 'eye' fit for safety . . . for unreflected systems", as it includes allowances for experimental and theoretical uncertainties. It is sufficiently conservative for use under any conditions where

*Similar calculations for infinite slabs give results which are not appreciably different and only one curve is shown for both.

significant neutron reflection of system components is virtually impossible. "However, for cases where the calculated $k$ is greater than 0·9 it is suggested that interaction specifications depend upon direct experimental data."

When the value of $k$ for a given container has been determined, the maximum permissible solid angle subtended at the most central body in an array of such containers is obtained from Curve A. The spacing of the containers is then adjusted so that this value is not exceeded. Several worked examples are given in reference (10), and Nicholls *et al.*[14] give values of general interest in the range covered by the experimental data. These values are reproduced below in Tables 2 and 3.

TABLE 2

k-*Values for Bare Cylinders of* $UO_2F_2$ *Aqueous Solution* (93% *Assay*)

| Diameter (in.) | Height (in.) | $H/U^{235}$ ratio | kg of $U^{235}$ per litre | $k$ |
|---|---|---|---|---|
| 5 | 12 | 50 | 0·48 | 0·52 |
| 5 | 24 | 50 | 0·48 | 0·55 |
| 5 | ∞ | 50 | 0·48 | 0·57 |
| 5 | 12 | 150 | 0·17 | 0·47 |
| 5 | 24 | 150 | 0·17 | 0·50 |
| 5 | ∞ | 150 | 0·17 | 0·52 |
| 5 | 12 | 350 | 0·075 | 0·39 |
| 5 | 24 | 350 | 0·075 | 0·42 |
| 5 | ∞ | 350 | 0·075 | 0·43 |
| 8 | 6 | 50 | 0·48 | 0·68 |
| 8 | 6 | 150 | 0·17 | 0·63 |
| 8 | 12 | 150 | 0·17 | 0·81 |
| 8 | 6 | 350 | 0·075 | 0·54 |
| 8 | 12 | 350 | 0·075 | 0·70 |
| 8 | 24 | 350 | 0·075 | 0·77 |
| 12 | 6 | 150 | 0·17 | 0·83 |
| 12 | 6 | 350 | 0·075 | 0·72 |

TABLE 3

*Maximum* k-*Values for* $UO_2F_2$ *Aqueous Solution*

| System | Maximum $k$ |
|---|---|
| 5 in. diameter cylinder | 0·58 |
| 8 in. diameter sphere | 0·67 |
| 1·25 in. thick slab | 0·24 |
| 350 g $U^{235}$ in 11·4 in. dia. sphere (volume 12·5 litres) | 0·65 |

3.3 *Effects of Full and Partial Reflection*

ORGDP philosophy on the effect of neutron reflectors upon units in an array is summarized in criteria (i) and (ii) at the beginning of this section. Water is a good reflector and is the only one that need be of concern to the ORGDP, accidentally or otherwise. Experiments showing that vessels which are separated by one foot of water are essentially isolated are quoted in support of criteria (i) and (ii). This is also apparent from the Los Alamos data for two spheres in water reproduced here in Fig. 2.

However, it is possible for two containers to be partially reflected and yet have no neutron absorbing material between them. The interaction in such a system is obviously greater than in either the bare or fully flooded systems at separations of one foot or more. The worst case of such partial reflection is when a thick reflector is placed on the outside half of each of the pair of units. In this case the value of $k$ for each partially reflected container ($k_{\frac{1}{2}}$) would be expected to lie roughly half-way between the fully bare and the fully reflected value, i.e.

$$k_{\frac{1}{2}} \approx \tfrac{1}{2}(k_{\text{bare}} + k_{\text{refl}})$$

Again the worst case here is when the fully reflected unit is just critical in its own right, a condition that would always be avoided in setting up the array. A safe, effective, value of $k$ to be used for arrays where partial reflection is possible is therefore,

$$k_{\frac{1}{2}} = \tfrac{1}{2}(k_{\text{bare}} + 1) \tag{3.1}$$

Experiments that have been done at ORNL indicate that evaluating $k_{\frac{1}{2}}$ from (3.1) is conservative even when the container is critical with full reflection.

The application of equation (3.1) to Curve A of Fig. 4, the safe curve for bare containers, produces the curve in Fig. 4 labelled Curve B. From what has been stated above, this curve should give an adequate safety curve for partially reflected systems, and, if the criterion of one foot minimum spacing is also observed, should suffice for any amount of reflection.

3.4 *Interaction Between Dissimilar Units*

Reference (10) mentions, without details, experiments that have been carried out with pairs of interacting units which are dissimilar in shape and content. The results seem to indicate that if unit $A$ is safe at distance $d_A$ from a similar unit, and another unit $B$ is safe from its twin at distance $d_B$, then the dissimilar units $A$ and $B$ are safe when separated at distance $(d_A + d_B)/2$. A justification for this rule when separation distances are reasonably large is presented in Section 5.3 and a generalization is given for smaller separations.

It is also stated that there appears to be no appreciable effect upon interaction due to differences in assay or moderation in the types of dissimilar systems studied.

3.5 *Other Factors Affecting Interaction in the ORGDP Method*

The ORGDP safety criteria laid down so far are based upon interaction experiments with highly and well-moderated fissile materials. At a given $k$-value a low-enrichment unit is less sensitive to changes in external neutron sources than is a high-enrichment unit, so that curves A and B of Fig. 4 are more conservative for low-enrichments than for high-enrichments. Also, from a practical point of view, conditions of half-reflection are less likely to be achieved with the physically larger low-enrichment units.

For poorly-moderated units (arbitrarily defined as those with $H/U^{235}$ ratios less than 20) no such general ruling can be given owing to the scarcity of experimental data on such systems. It would therefore seem to be unwise to apply the ORGDP method to poorly-moderated systems until further checks have been carried out.

It is possible for the uranium solution concentration to change during storage, for example, by precipitation. To allow for this contingency Henry *et al.*[9] have calculated the change in $k$-value with volume for a bare sphere containing the minimum "safe" mass of $U^{235}$ (350 g). They find a maximum $k$-value of 0·65 at 12·5 l. Hence, in dealing with containers in which this effect can occur this maximum $k$-value should be used when evaluating the safe solid angle from Fig. 4.

3.6 *Summary of the Method and Formulae for k*

An array of identical solution containers is considered to be safe if the following criteria are satisfied.

(1) That all containers are safe individually when completely reflected by water.
(2) The container separations are never less than 12 in.
(3) The total solid angle subtended at the most central unit by all other units is less than the following values:

$$0{\cdot}48 \times 4\pi \quad \text{for} \quad k < 0{\cdot}3$$

$$(0{\cdot}72 - 0{\cdot}80k) \times 4\pi \quad \text{for} \quad 0{\cdot}3 < k < 0{\cdot}8$$

Containers which are shielded from the most central unit are not included in the total solid angle.

(4) For $k > 0{\cdot}8$, the separation should be based on direct experimental data.
(5) A solid angle of 0·04% of $4\pi$ may be neglected when considering the interaction between individually safe units.

Values of $k$ for use in the above criteria are calculated from the following formulae[10]. They apply only to highly enriched, well-moderated, bare systems.

$$k = \eta f U_f U_t, \tag{3.2}$$

where

$\eta$ = average number of fast neutrons produced per thermal capture in $U^{235}$.

$f$ = probability of capture in $U^{235}$ (thermal utilization factor).

$U_f$ = probability that a fast neutron does not escape from the system before thermalization (non-leakage probability for fast neutrons).

$U_t$ = probability that a thermal neutron does not escape from the system before capture (non-leakage probability for thermal neutrons).

For low-assay systems the resonance escape probability[10] must also be included in (3.2).

In highly enriched uranium $\eta$ is constant with the value 2·09[10]. The factor $f$ is determined from the ratio at thermal energies of the absorption cross section of $U^{235}$ to the total absorption cross section. When the only absorber present in addition to $U^{235}$ is hydrogen (as for example with $UO_2F_2$ solutions) the value of $f$ is given by

$$f = \frac{1}{1 + 4{\cdot}8 \times 10^{-4} \times (\mathrm{H/U^{235}})} \tag{3.3}$$

The fast non-leakage probability, $U_f$, can be obtained from the following empirical relation:

$$U_f = \frac{1}{(1 + B_1^2)(1 + 4{\cdot}2B_1^2)(1 + 20{\cdot}16B_1^2)} \tag{3.4}$$

where $B_1{}^2$ is the geometric buckling (in $cm^{-2}$) of the system for fast neutrons. This is obtained from the usual formulae, demonstrated in Table 4 below, with an extrapolation length, $\lambda$, of 2·5 cm.

TABLE 4

*Geometric Bucklings*

| System | Buckling ($B^2$) |
|---|---|
| Plane Slab, thickness $T$ | $\dfrac{\pi^2}{(T + 2\lambda)^2}$ |
| Sphere, radius $R$ | $\dfrac{\pi^2}{(R + \lambda)^2}$ |
| Cylinder, height $H$ diameter $D$ | $\dfrac{\pi^2}{(H + 2\lambda)^2} + \dfrac{4{\cdot}81^2}{(D + 2\lambda)^2}$ |

The two-group formula (3.4) is quoted by Knight[10] as giving good agreement with experiment for 90% assay solutions with $H/U^{235}$ atomic ratios greater than 40. It is also usable at lower assays with a proper value for the resonance escape probability included.

The thermal non-leakage probability, $U_t$, is given by

$$U_t = \frac{1}{1 + B_2^2 L^2} \tag{3.5}$$

where $B_2^2$ is the geometric buckling ($cm^{-2}$) for thermal neutrons and $L^2$, the square of the diffusion length, is given by

$$L^2 = 8{\cdot}29(1 - f), \quad (cm^2). \tag{3.6}$$

$B_2^2$ is obtained in exactly the same manner as $B_1^2$ except that an extrapolation length of 0·36 cm is to be used in place of 2·5 cm.

Values of $k$ can, of course, be calculated by more accurate means than the set of equations (3.2, 3, 4, 5, 6) given above. However, it must first be checked that the two-group values of $k$ are not significantly larger than the more accurate values for then the use of Fig. 4 (designed to fit the two-group values) may give results erring on the dangerous side.

The check calculations that have been done by Henry *et al.*[9] refer to poorly moderated systems of low assay, for it is with such systems that the two-group recipe gives poorest results. Table 5 below gives a brief summary of the results for bare containers of $UO_2F_2$ which were experimentally measured to be critical, i.e. the exact value of $k$ is unity.

TABLE 5

*Comparison of Two-group Recipe with Multigroup Diffusion Theory for Critical Systems*

| $U^{235}$ Assay | $H/U^{235}$ Ratio | $k$ | Calculation method |
|---|---|---|---|
| 37·5% | 0·1 | 1·0491 | Multi-group |
| 37·5% | 5·1 | 1·1995 | Multi-group |
| 30·0% | 32·0 | 1·0414 | Multi-group |
| 30·0% | 32·0 | 1·1368 | 2-group |

In the above cases the calculations are conservative, the two-group approximation being more conservative than the more accurate multigroup approach. However, this does not mean that a multi-group approach which does not employ the diffusion approximation (e.g. Carlson's $S_n$ method[16]) will also be conservative. Also no check has been made for systems with small values of $k$, although in this region Fig. 4 indicates that on Henry's criterion a fairly large error in $k$ can be tolerated without hazard. One of us (R.A.S.) has checked that the one-group form of (3.2) is conservative for all $k < 1$ when compared with one-group Carlson–$S_4$ calculations on bare Oy(93·5) spheres of various sizes up to the critical size.

For practical purposes it would seem advisable to employ the simple two-group recipe set out above and to keep to the regions where it is known to be reliable.

## 4. THE INTERACTION PARAMETER METHOD

This method was first used at AWRE in 1954 by one of us (A.F.T.) when considering arrays of metal spheres, and has been applied elsewhere by Mayne[15] and Woodcock[14]. It is designed to provide a measure of the degree of criticality of an array of fissile bodies in terms of a quantity which is characteristic of each body and which is capable of experimental measurement with single bodies or of deduction from other experimental data. A practical method of this type was required at that time since the calculating methods then in existence were not considered reliable enough for safety purposes. Present techniques, using fast computers, are bringing theory more into line with experiment and nowadays calculated values of the parameter can be used with more confidence.

The "interaction parameter", $q_{ij}$, for one body ($i$) as viewed from another body in the array ($j$) is defined as the total neutron output induced in $i$ when one neutron leaves $j$ and does not interact with any other bodies in transit. Alternatively, if $F_j$ is the total neutron output of body $j$ when situated in the array then the output induced in body $i$ due to body $j$ alone is $q_{ij} F_j$.

As defined in this way $q_{ij}$ can be represented as the product of the probability, $p_{ij}$, that a neutron leaves body $j$ and reaches body $i$ without interacting with any other bodies on the way, and the surface multiplication, $M_{si}$, of body $i$ to neutrons reaching it from $j$. The quantity $p_{ij}$ will in general depend upon

(i) the shape of body $j$ and the angular distribution and energy spectrum of neutrons leaving $j$;

(ii) the separation between $i$ and $j$ and the neutronic properties of the intervening medium;

(iii) the amount of screening presented by other bodies of the array in transit from $j$ to $i$.

The quantity $M_{si}$ is governed entirely by the multiplication properties of body $i$ for neutrons reaching it from $j$, due account being taken of the fact that neutrons induced in $i$ may return to $i$ for subsequent multiplication without having interacted with other bodies in the array.

In practice it is almost certain that exact values of $q_{ij}$, implying a full knowledge of the factors (i), (ii) and (iii) above, will not be available except for simple two-body systems. Here in principle the necessary measurements can be made, but values of $q_{ij}$ obtained from two-body experiments are not exactly the same as those which obtain when the remainder of an array is brought into place, even ignoring screening effects, for the consequent change in angular distribution of emission of the source body and a possible change of spectrum will play their part.

Hence in translating from the quantity $q_{ij}$ defined above to a practical parameter to be used in clearing arrays these effects of angular distribution and spectrum must be simulated in such a way as to overestimate the required interaction. In what follows the theory will be set up ignoring these practical difficulties. Means of overcoming them will be mentioned in the sections devoted to assigning experimental or theoretical values to the parameter $q_{ij}$.

4.1 *Application of the Method to Arrays*

The use of the parameter $q_{ij}$ in establishing the safety of arrays of fissile bodies is described below, firstly for the case of two interacting bodies and then for a multi-body array.

4.1·1 *The two-body problem.*—Consider two fissile bodies whose steady neutron outputs in isolation are $F_1$ and $F_2$, respectively. Suppose that when placed at some finite distance apart these outputs rise to values $F'_1$ and $F'_2$. From the definition of the interaction parameter the number of neutrons induced in body 1 as a result of the presence of body 2 is $q_{12}F'_2$, and so

$$F'_1 = F_1 + q_{12}F'_2. \tag{4.1}$$

Similarly

$$F'_2 = F_2 + q_{21}F'_1. \tag{4.2}$$

Hence

$$F'_1 = \frac{F_1 + q_{12}F_2}{1 - q_{12}q_{21}}, \quad F'_2 = \frac{F_2 + q_{21}F_1}{1 - q_{12}q_{21}} \tag{4.3}$$

From (4.3), subcritical states correspond to cases where

$$q_{12}q_{21} < 1. \tag{4.4}$$

In the critical state $q_{12}\,q_{21}$ is equal to unity, for then finite values of $F'_1$ and $F'_2$ can be maintained in the absence of neutron sources.

For two like bodies

$$F'_1 = F'_2 = F', \quad F_1 = F_2 = F, \quad \text{and} \quad q_{12} = q_{21} = q,$$

and equations (4.3) reduce to

$$F' = F/(1 - q) \tag{4.5}$$

Some idea of the way in which the value of $q$ varies with separation for two 20 kgm oralloy spheres in water can be obtained by associating the reciprocal multiplication in Fig. 2 with $F/F'$, i.e. with $(1–q)$ from (4.5).

4.1.2 *The n-body problem.*—The generalization of equations (4.1.2) to the case of $n$ bodies is immediate. Let $F_i$, $F'_i$ be the outputs of the $i$th body in isolation and in array, respectively.

Then

$$F_1' = F_1 + q_{12}F_2' + q_{13}F_3' + \ldots + q_{1n}F_n',$$

$$F_2' = q_{21}F_1' + F_2 + q_{23}F_3' + \ldots + q_{2n}F_n',$$

$$F_3' = q_{31}F_1' + q_{32}F_2' + F_3 + \ldots + q_{3n}F_n',$$

and so on.
The condition for criticality is now

$$D \equiv \begin{vmatrix} -1 & q_{12} & q_{13} & \ldots \\ q_{21} & -1 & q_{23} & \ldots \\ q_{31} & q_{32} & -1 & \ldots \\ \vdots & \vdots & & \\ q_{n1} & q_{n2} & \ldots & -1 \end{vmatrix} = 0 \tag{4.6}$$

Equation (4.6) will be referred to as the "critical equation".

Hence in principle if all the appropriate two-body interactions, $q_{ij}$, are known the degree of criticality of the array can be assessed. In practice, especially with large arrays, the solution of (4.6) to give the critical conditions can be tedious and it is usually sufficient to make use of approximate solutions.

4.1.3 *Approximate but safe solutions of the critical equation.*—As a rough guide one can replace all values of $q_{ij}$ in (4.6) by the maximum value that occurs, call it $q_{\max}$. This will obviously underestimate the safe number in the array. Then

$$D = (-1)^{n-1}(1 + q_{\max})^{n-1}[(n-1)q_{\max} - 1]$$

and so for criticality

$$(n-1)q_{\max} = 1. \tag{4.7}$$

Equation (4.7) is oftcn too conservative to apply in practice and some refinement is needed. This can be carried out in the following manner.

At criticality the equations for the neutron outputs take the form

$$F_i' = \sum_j{}' q_{ij}F_j', i \quad = 1, 2, \ldots, n \tag{4.8}$$

where the prime attached to the summation sign indicates that the term with $i = j$ is omitted. Suppose that the largest output comes from the body with $i = x$. Then from (4.8)

$$F_i' \leqq F_x'S_i, \quad S_i \equiv \sum_j{}' q_{ij}, \tag{4.9}$$

and in particular

$$F'_x \leqq F'_x S_x,$$

i.e.

$$S_x \geqq 1. \tag{4.10}$$

Hence for criticality at least one of the quantities $S_i$ must exceed unity, and so the condition

$$S_{\max} \equiv \max_i \{S_i\} < 1 \tag{4.11}$$

is sufficient to ensure that the array is subcritical.

For a symmetrical array of like bodies the maximum value of $S_i$ will occur for the most centrally situated body ($x$) and the $q_{xj}$ will all be proportional to the interaction parameter ($q$) at unit lattice spacing. Equation (4.11) then provides a safe upper limit to the value of $q$.

For an infinite symmetrical array all the $S_i$ are equal (to S, say) and $S < 1$ again ensures subcriticality.

In most cases of practical interest equation (4.11) is sufficient to give reasonably accurate results. For symmetric arrays of like bodies where the body with maximum output is known the solution can be carried on iteratively in the following way.

Substitute equation (4.9) into (4.8) to get

$$F'_j \leqq F'_x S_i^{(1)}, \quad S_i^{(1)} \equiv \sum_j{}' q_{ij} S_j$$

which leads to (4.11) being replaced by

$$S_{\max}^{(1)} \equiv \max_i \{S_i^{(1)}\} < 1, \tag{4.12}$$

and so on to higher approximations.

Table 6 below gives the values for the maximum interaction parameter at unit lattice spacing obtained from these successive approximations when the interaction is assumed to fall off inversely as the distance in various arrays of like bodies, the shielding of one body by another not being allowed.

TABLE 6

*Successive Approximations to the Maximum-q Value at Unit Spacing*

| Array | $q=1/n-1$ | $S_{\max}=1$ | $S_{\max}{}^{(1)}=1$ | $S_{\max}{}^{(2)}=1$ | Complete solution |
|---|---|---|---|---|---|
| Square 3 × 3 | 0·1250 | 0·1464 | 0·1638 | 0·1700 | 0·1786 |
| Square 4 × 4 | 0·0667 | 0·1003 | 0·1099 | 0·1133 | 0·1169 |
| "Hexagonal" 7 | 0·1667 | 0·1667 | 0·1892 | 0·1962 | 0·2048 |
| "Hexagonal" 19 | 0·0556 | 0·0802 | 0·0894 | 0·0932 | 0·0999 |
| Cubic 27 | 0·0385 | 0·0523 | 0·0586 | 0·0614 | 0·0652 |

In the above cases the difference between the solution obtained from (4.11) and the true solution is never more than 20%, erring on the safe side. When considering general asymmetrical arrays, the extra computational effort required to obtain a more accurate solution than that given by (4.11) is rarely justified.

When the most reactive body or bodies in an array are surrounded by a large number of much smaller bodies it is often useful to work in terms of the complementary condition to (4.11), namely

$$T_{\max} \equiv \max_j \{\sum_i{}' q_{ij}\} < 1. \tag{4.13}$$

This relation can be obtained directly from (4.8) by summing these equations over $i$ and rearranging from a sum by rows to a sum by columns. In this way

$$\sum_i F'_i = \sum_i \{\sum_j{}' q_{ij} F'_j\} = \sum_j \{\sum_i{}' q_{ij}\} F'_j,$$

so that

$$\sum_j (-1 + T_j) F'_j = 0, \quad T_j = \sum_i{}' q_{ij}.$$

The outputs $F'_j$ are all necessarily positive, so that in general some of the factors $T_j$ will be greater than unity and the remainder less than unity. In particular $T_{\max}$ as defined by (4.13) will be greater than unity in the critical system. The criterion that $T_{\max}$ should not exceed unity for any given array therefore ensures that the array is subcritical.

From the definition of $q_{ij}$ it is obvious that the factors $S_i$ and $T_j$ represent effective multiplication factors for the array as a whole. $S_i$ gives the total number of neutrons induced in body $i$ when one neutron leaves each other body of the array but does not react with bodies other than body $i$ on the way, and $T_j$ gives the total number of neutrons produced in the same way in all other bodies of the array when one neutron leaves body $j$.

4.1.4 *General points on the evaluation of $S_i$ and $T_j$.*—To apply (4.11) or (4.13) to practical arrays, the values of $S_i$ or $T_j$ are calculated for all bodies in the array (unless it is obvious where their maxima lie). If the maximum value occurring is less than unity the array is safe. If the maximum value is slightly greater than unity it may be worthwhile going to a higher approximation or even directly evaluating the determinant in (4.6) if this is feasible. However, in most cases this event would be an indication that the array was unsafe and alterations to the array in the form of increasing the lattice pitch, decreasing the number of bodies, or the introduction of neutron absorbers, necessary.

The way in which the effect of the shielding of one body by another is introduced into the value $q_{ij}$ for estimating $S_i$ to $T_j$ is left to the discretion of the user. It has been the practice at AWRE to completely ignore shielding in order to allow flexibility in the conditions imposed by the resulting criticality clearances. For example, in workshops handling fissile material the positions are not

immutable and line-of-sight shielding could not be assumed without enforcing considerable restrictions upon freedom of movement. (In the technique of Henry *et al.* described in the last section line-of-sight shielding in air-spaced lattices is allowed.)

Further, it has been AWRE practice to assume even when dealing with air-spaced lattices that the value of $q_{ij}$ falls off inversely as the first power of the separation distance [see equation (4.22)] rather than the second power at large enough separations. This assumption will always be conservative and is intended to overestimate the effect of such scattering objects as walls and floors of buildings.

4.2 *Experimental Estimation of q*

Since individual units of arrays must themselves be subcritical under all reasonable circumstances of damage and accidental reflection they tend in fact to be well subcritical and hence have only moderate neutron multiplications. Values of $q$ for such bodies are often small ($\approx 10^{-4}$–$10^{-2}$) and difficult to measure with accuracy. Thus measurements are usually made with the fissile body and a neutron source* rather than with the two fissile bodies in question. This raises the problem of how best to simulate the second fissile body by means of a small neutron source so as to ensure that the value of $q$ obtained is either accurate or at least errs on the side of safety.

4.2.1 *The experimental method.*—The principle of the method of measurement is as follows

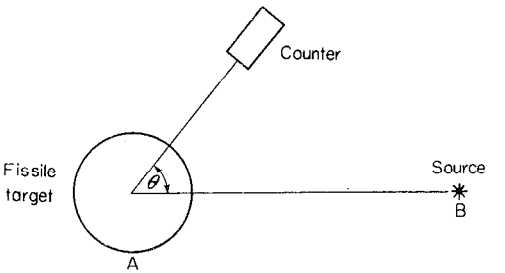


FIG. 5.

To measure the output induced in a fissile target at $A$ (see Fig. 5) due to a source at $B$ the following count rates are measured.

(1) Source at $A$ alone ($C_1$)
(2) Fissile body at $A$ alone ($C_2$)
(3) Source at $B$ alone ($C_3$)
(4) Fissile body at $A$, source at $B$ ($C_4$)

*Usually small mock fission sources, Po—B, F, Be, Na ($\alpha$, $n$), with outputs of about $10^6$ neutrons/sec are used.

Then if $Q$ is the neutron output of the fissile body in isolation, $S$ is the source strength, and $E_1$, $E_2$ the counter efficiencies for the source at $A$ and $B$, respectively, then:

$$C_1 = E_1 S,$$

$$C_2 = E_1 Q,$$

$$C_3 = E_2 S,$$

$$C_4 = E_1(Q + qS) + E_2 S,$$

$q$ being the required interaction parameter.

Thus
$$q = \frac{C_4 - (C_2 + C_3)}{C_1} \tag{4.14}$$

For $q$ as obtained from (4.14) to be a good approximation to the required $q$-value for two bodies the following conditions should be fulfilled.

(a) That the source simulates as closely as possible the neutron emission of the body it replaces.

(b) The counter efficiencies for neutrons leaving the target body at position $A$, and the source at position $A$ should be equal.

(c) The counter position should be such that none of the neutrons which contribute to the count $C_3$ should be absorbed or deflected by the target body in count $C_4$.

As stated earlier, condition (a) cannot always be realized and the best that can be done is to arrange the position of source and counter so that an overestimate of $q$ is obtained. Then, whenever possible, corrections are applied (conservatively) to improve the value.

4.2.2 *Application of the method.*—A series of measurements on the interaction parameter for uranium cylinders of 93% $U^{235}$ alone and diluted with natural uranium and graphite, illustrates the technique.

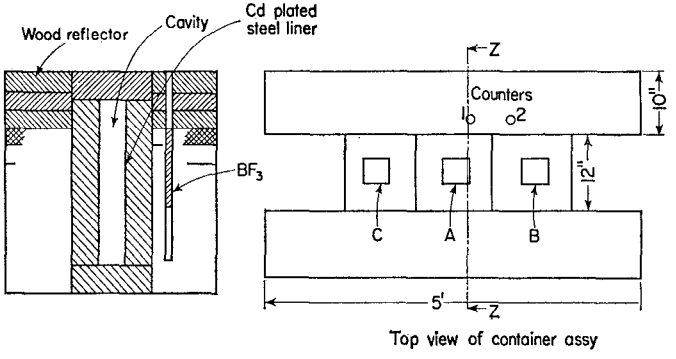


FIG. 6. Experimental arrangement for measuring the interaction between transit containers.

The value of $q$ was required for 2·9 in. diameter cylinders of uranium of varying enrichments and highly enriched uranium mixed with graphite in wooden transit containers used for the transport and storage of the metal and of broken crucible pieces. The containers are of teak with dimensions, 12 in. × 12 in. × 34 in., and have a central cavity 4 in. × 4 in. × 26 in., cadmium lined. The experimental arrangements are shown in elevation and plan in Fig. 6.

The value of $q$ as given by equation (4.14) was measured for a series of cylinders of solid 93% $U^{235}$, and for cylinders of 93% $U^{235}$ and natural uranium or graphite made up of interleaved $\frac{3}{8}$ in. thick discs. The source used was a mock fission source [Po, — B, F, Be, Na ($\alpha$, n)] and was placed half-way up the cavity in container $B$ on the wall nearest container $A$ for counts $C_3$ and $C_4$ and at the centre of the cavity of container $A$ for count $C_1$.

The value of $q$ so measured was not the value appropriate to an average neutron leaving a cylinder in container $B$, since

(a) the efficiency of the counters for a source at the centre of container $A$ was greater than for a source distributed over the target cylinder;

(b) the fraction of the neutrons leaving a source at $P$ which strike the fissile target was greater than that for a source appropriately distributed over a similar source cylinder;

(c) the distribution of the re-emitted neutrons from the target cylinder was more peaked towards the centre with a source at $P$ than with a distributed source in container $B$, thus giving too high a count rate for count $C_4$.

Of these effects, (a) would tend to make the value of $q$ obtained an underestimate, whilst (b) and (c) will give an overestimate.

Subsidiary experiments were performed to determine (i) the variation of count rate on both counters with source position in container $A$, (ii) the variation in the count rate of a $U^{235}$ fission counter with position in container $A$ for a source at position $P$, (iii) the variation in multiplication of the cylinder of fissile material with the position of a source on its axis.

If $x$ and $y$ denote distances of points $X$ and $Y$ from the centres $A$ and $B$ respectively, and:

$E(x)$ = relative efficiency of the counters for a source at $X$

$p(x, y)$ = relative probability that a neutron leaving $Y$ will strike the target cylinder at $X$

$M(x)$, $M(y)$ = Multiplication of the fissile cylinder for a source at positions $X$ and $Y$

Then

$$q = K \times q\,(\text{apparent})$$

where

$$K = \frac{\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} E(x)M(x)p(x,y)M(y)\,dx\,dy}{\int_{-L/2}^{L/2} M(y)\,dy \int_{-L/2}^{L/2} E(x)M(x)p(x,o)\,dx}$$

The values of $K$ were evaluated for the 9 in. and 18 in. cylinders, using the value of $p(L/2, O)$ for $(x - y) > L/2$ (overestimating $K$). The values obtained were as follows:

TABLE 7

*Values of K*

| Target | Counter 1 | Counter 2 |
|---|---|---|
| 9 in. cylinder (18 kg $U^{235}$) | 0·710 | 0·692 |
| 18 in. cylinder (36 kg $U^{235}$) | 0·869 | 0·856 |

The resulting values of $q$ are listed in Table 8.

In experiment 8, the target cylinder was placed in the left-hand container (Fig. 6) so that the value of $q$ is appropriate to that for containers separated by a third container. No correction has been applied to this result; it would be expected to be intermediate between that for the 9 in. (18 kg) and 18 in. (36 kg) cylinders in adjacent containers.

TABLE 8

*Interaction Parameters for Transit Containers*

| Experiment | Source position | Target | $q$ (Uncorrected) $\times 10^2$ | | $q \times 10^2$ (Corrected mean) |
|---|---|---|---|---|---|
| | | | Counter 1 | Counter 2 | |
| 1 | (a) | 36 kg 93% $U^{235}$ | 4·07 ± 0.02 | 4·01 ± 0·03 | 2·85 ± 0.02 |
| 2 | (a) | 18 kg 93% $U^{235}$ | 2·27 ± 0·03 | 2·31 ± 0·04 | 1·97 ± 0·03 |
| | (b) | 18 kg 93% $U^{235}$ | 1·85 ± 0·03 | 1·76 ± 0·04 | 1·82 ± 0 03 |
| 3 | (a) | 9 kg $U^{235}$ + C (25% U by vol.) | 1·09 ± 0·04 | 0·92 ± 0·10 | 0·73 ± 0·03 |
| 4 | (a) | 4½ kg $U^{235}$ + C (12½% U by vol.) | 0·44 ± 0·04 | 0·34 ± 0·06 | 0·30 ± 0·03 |
| 5 | (a) | 18 kg $U^{235}$ 18 kg Nat U | 1·85 ± 0·03 | 1·92 ± 0·03 | 1·32 ± 0·02 |
| 6 | (a) | 27 kg $U^{235}$ 9 kg Nat U | 2·65 ± 0·04 | 2·67 ± 0·04 | 1·86 ± 0·03 |
| 7 | (a) | 18 kg $U^{235}$ (Cd liner removed) | 6·11 ± 0·05 | 6·58 ± 0·05 | — |
| 8 | (a) | 36 kg $U^{235}$ | 0·285±0·108 | 0·247±0·020 | — |

Source position—(a) half-way up cavity in container $B$ at wall nearest $A$.
(b) centre of 18 kg $U^{235}$ cylinder in container $A$ to simulate a distributed source.

4.2.3 *Application of the results to arrays of transit containers.*—The results given in Table 8 indicate that dilution of a given mass of $U^{235}$ by graphite or natural uranium reduces the interaction parameter. If assumptions are made

about the variation of the parameter with the centre-to-centre separation ($d$) of the containers deductions can be made as to the safety of arrays of loaded containers. In these calculations take $q_0$, the interaction parameter for 36 kg $U^{235}$ cylinders in adjacent containers, to be 0·0293 (mean value + 4 standard deviations).

(i) Assume that $q \propto 1/d$. This will overestimate the value of $q$ for large separations as stated earlier.

For $N$ loaded containers in a rectangular plane lattice it can be shown that

$$S_{\max} < 3{\cdot}525(\sqrt{N} - 1)q_0 \tag{4.15}$$

for the most reactive arrangement*.

Combining condition (4.11) with (4.15) above gives

$$N < 114$$

as a sufficient condition for subcriticality.

For three-dimensional parallelepiped stacking of the containers it is reasonable to assume that the interaction parameter for non-coplanar containers is equal to that for coplanar containers at the same separation.

Then proceeding as above it is found that

$$N < 60$$

for safety.

(ii) Now assume that $q \propto e^{-\alpha d}/d$.

This is a more plausible assumption, for this type of variation is indicated by one-group diffusion theory if $\alpha$ is associated with the diffusion length in the reflecting material (see equation 4.22)). For brevity write $\mu$ for the attenuation factor in the wood (function of $\alpha$). Then on the same assumptions as used above it can be shown that the safety criterion for the infinite three-dimensional lattice is

$$\frac{19{\cdot}1}{(1-\mu)^2} q_0 < 1.$$

For the array of fully loaded containers this condition is satisfied if $\mu$ is less than 0·252. This in turn is true if the value of $q$ obtained in Expt. 8 is less than 0·126 of the value obtained in Expt. 1, i.e. provided

$$q(8) < 0{\cdot}359 \pm 0{\cdot}003 \times 10^{-2}$$

Thus, in view of the very conservative assumptions made about the interaction between non-coplanar containers, the above results indicate very strongly that an infinite array of loaded containers would be subcritical.

*This result was derived independently by Mayne[15].

4.2.4 *Other experimental results.*—Some other experimental results of general interest for metal spheres are given in Table 9 below.

TABLE 9

*Interaction Parameters for Metal Spheres at* 2ft 6 in. *Centre-to-Source Separation*

| Target | Intervening medium | Interaction parameter $q \times 10^4$ | | $q \times 4\pi/\Omega.M_c$ |
|---|---|---|---|---|
| (a) 20 kg 93% $U^{235}$ | Air (6 ft from ground) | 58·4 ± 3·9 | (90°) | 1·05 ± 0·07 |
| | | 51·7 ± 3·3 | (130°) | 0·93 ± 0·06 |
| (b) 20 kg 93% $U^{235}$ | Air (Container on ground) | 71·3 ± 2·3 | (40°) | 1·28 ± 0·04 |
| | | 70·6 ± 1·2 | (90°) | 1·27 ± 0·02 |
| | | 54·6 ± 1·1 | (135°) | 0·98 ± 0·02 |
| (c) 20 kg 93% $U^{235}$ | Sand–Cd covered | 4·5 ± 0·4 | (90°) | 0·08 ± 0·01 |
| | Uncovered | 29·0 ± 0·4 | (90°) | 0·52 ± 0·07 |
| (d) 20 kg 93% $U^{235}$ | Water–Cd covered | 0·09 ± 0·01 | (90°) | $1·6 \times 10^{-3}$ |
| | Uncovered | 0·37 ± 0·02 | (90°) | $7·0 \times 10^{-3}$ |
| (e) 9 kg Pu (15·6 gm/cc) | Air (Container on ground) | 83 ± 6 | (90°) | 1·44 ± 0·10 |
| | | 76 ± 6 | (135°) | 1·33 ± 0·10 |
| (f) 9 kg Pu | Water–Cd covered | 1·2 ± 1·2 | (90°) | 0·02 ± 0·02 |
| (g) 7·6 kg Pu | Air (Container on ground) | 60 ± 4 | (90°) | 1·48 ± 0·10 |
| (h) 4·6 kg Pu | Air Container on ground) | 29 ± 3 | (90°) | 1·58 ± 0·16 |

(i) Angles given in parentheses refer to the angle $\theta$ in Fig. 5.
(ii) In experiments (b) to (h) the source and the target sphere were at the centre of steel containers, 24 in. high and 12 in. diameter; the steel thickness was $\frac{1}{10}$ in.
(iii) In experiments (c) the containers stood on the ground and were buried in sand with a minimum thickness of 1 ft in all directions.
(iv) In experiments (d) and (f) the containers stood in a tank and were flooded to a level of 1 ft above the containers.

Column 4 lists values of $q$ divided by the product of the solid angle fraction subtended by the sphere at 2 ft 6 in. and the central source multiplication, $M_c$, of the sphere. This quantity is thus a measure of the effect of scattering and absorption in the surrounding medium, the value of this quantity for air-spaced systems being always of order unity. The importance of a cadmium shield is well illustrated in experiments (c) and (d).

4.3 *Estimation of Interaction Parameters from Multiplication Measurements*

In certain cases the direct measurement of an interaction parameter involves long and tedious experiments. However, it is usually sufficient to show that the $q$-value in question is less than the optimum value required to ensure the safety of the array and approximate but conservative $q$-values can be obtained from experimental values of central source multiplications. If the latter are not already available they can be measured fairly simply.

The starting point in this method is the equation mentioned at the beginning of Section 4, namely

$$q = pM_s \tag{4.16}$$

where the suffices $i$ and $j$ have been dropped for brevity. In describing methods of estimating $p$, the transmission probability, and $M_s$, the surface multiplication, it is convenient to distinguish between bare and reflected systems. Furthermore, it will be implied that these systems have spherical symmetry since, apart from other considerations of simplicity, non-spherical shapes are less reactive and have values of $M_s$ which vary over the surface.

4.3.1 *Bare Systems.*—For a bare fissile core the value of $p$ is given simply by the mean solid angle fraction subtended by the core at the second body. The word "mean" here is used in the same sense as used by Henry *et al.* (see Section 3). However, some confusion can arise for closely spaced bodies and for safety purposes the solid angle fraction subtended at the nearest point of the second body is usually employed.

To estimate $M_s$ for a bare system, call it $M_{sb}$, this quantity must first be related to the measured multiplication, i.e. the central source multiplication, defined as the number of neutrons emerging from the sphere when a unit isotropic source is placed at the centre.

To do this consider an isotropic point source of strength $Q$ placed on the surface of the system. Each ingoing neutron, $\frac{1}{2}Q$ in number, will give rise to $M_{sb}$ outgoing neutrons; the $\frac{1}{2}Q$ outgoing neutrons do not return to the system. The total emission is therefore

$$\tfrac{1}{2}(M_{sb} + 1)Q.$$

If this same source is placed at the centre of the sphere the total emission would be $M_{cb}Q$. Hence we can write

$$\tfrac{1}{2}(M_{sb} + 1)Q = M_{cb}Q\Theta$$

i.e.

$$M_{sb} = 2\Theta M_{cb} - 1, \tag{4.17}$$

where $\Theta$ is a function which specifies the relative importance of neutrons injected at the outside to neutrons injected at the centre.

For systems whose dimensions are of the same order or smaller than one neutron mean free path there is no preferential point at which neutrons can be injected with peculiarly high multiplication, and for such systems $\Theta$, $M_{cb}$ and $M_{sb}$ will all be of order unity. For systems containing several mean free paths diffusion theory can be used as a guide. In this case it can be shown that the number of neutrons, $U(a, r)$, emerging from a sphere of radius "$a$" due to the presence of an isotropic spherical delta-function source of unit strength at radius $r$ ($< a$) is proportional to $(\sin Br)/(Br)$, where $B^2$ is the material buckling. Hence since

$$M(a, o) = M_{cb}, \qquad M(a, a) = \tfrac{1}{2}(M_{sb} + 1)$$

it follows from (4.17) that

$$\Theta(\text{diffusion theory}) = \sin Ba/(Ba) \tag{4.18}$$

Equation (4.18) has been checked against multi-group Monte Carlo results for bare metal spheres of various sizes and found to be quite accurate.

Thus for bare metal spheres there is sufficient evidence to indicate with some certainty that $\Theta$ is always less than unity, and consequently that, from (4.17):

$$M_{sb} \leqq 2M_{cb} - 1. \tag{4.19}$$

Intuition suggests that neutrons injected at the centre of a spherical system will, in general, be more effective than neutrons injected at the outside and hence that (4.19) is valid for any system. However, this has not yet been demonstrated, and a programme of calculations is at present under way at AWRE to test this thesis.

4.3.2 *Fully reflected systems.*—Two types of fully reflected system have to be distinguished, (i) where the whole lattice of fissile bodies is immersed in some continuous reflecting material (e.g. a fully flooded array), and (ii) where each fissile core is completely surrounded by a shell of reflecting material but the array of such units is air-spaced. In case (i) the factor $p$ involves both scattered and unscattered neutrons, and the requisite $M_s$ is equal to $M_{sb}$ multiplied by some factor to allow for reflection. In case (ii) $p$ is equal to the solid angle fraction subtended by the reflected unit at the point in question and $M_s$ now refers to the reflected unit.

To evaluate the factor $M_s$ for case (i) let $\epsilon_\infty$ be the number of neutrons which return to the fissile core from the surrounding medium for each neutron emitted. One neutron incident upon the core produces $M_{sb}$ outgoing neutrons; a fraction $\epsilon_\infty$ of these return to the core and each of these in turn will produce $M_{sb}$ outgoing neutrons, and so on. The total emission of the core per neutron incident on the surface is therefore

$$M_s = M_{sb} + \varepsilon_\infty M_{sb}^2 + \ldots = \frac{M_{sb}}{1 - \varepsilon_\infty M_{sb}} \tag{4.20}$$

It should of course be stressed that $M_{sb}$ in (4.20) refers to the energy spectrum of neutrons incident upon the core, a spectrum which in most practical instances will be softer than the emitted spectrum.

Combining equation (4.19, 20) gives an upper limit to $M_s$ in terms of $M_{cb}$ and $\epsilon_\infty$. In some cases the latter can be estimated directly from experimental data on neutron reflection. Where this does not exist an upper limit to $\epsilon_\infty$ can be obtained if it is possible to estimate the surface multiplication, $M_s'_b$, of the bare sphere of the material in question which is just critical when fully reflected, since

$$\varepsilon_\infty < 1/M'_{sb}$$

from (4.20). The "less than" rather than the "equal to" sign appears in this relation as the critical sphere will be larger than the sphere whose $q$-value is required and $\epsilon_\infty$ increases with the sphere radius.

If information on $M_{sb}$ is not available then some calculating method for obtaining $\epsilon_\infty$ directly must be used. It is not possible to generalize here in any way owing to the diversity of reflecting media that can occur, and each case must be treated on its merits. The simple one-group diffusion theory value of $\epsilon_\infty$ is sometimes of use; this takes the form

$$\varepsilon_\infty = \frac{1-\mu}{1+\mu}, \qquad \mu = \frac{2}{3}\left(\frac{1}{a}+\frac{1}{L_d}\right)L_t \tag{4.21}$$

where "$a$" is the radius of the sphere, and $L_d$ and $L_t$ are the diffusion length and transport mean free path, respectively, in the reflecting medium. Equation (4.21) has not been checked for conservativeness and so should only be used as a last resort, and then only in the cases where $L_d$ and $L_t$ can be meaningfully defined.

Similar comments apply to the evaluation of $p$, the remaining factor required to calculate $q$ from (4.16). If the necessary transmission data already exists or can be measured fairly readily then $p$ can be estimated. If not, then calculating methods must be resorted to, preferably along the lines indicated in Section 4.4. A simple imitation of these methods is to replace the (point) source by a spherical surface source at the same radius and then to calculate by conventional diffusion methods the number of neutrons per unit source strength which fall upon a completely "black" sphere of the same radius as the target, this being $p$. In the same manner as (4.21) was obtained and with the same notation the result of this calculation is

$$p = \frac{1}{1+\mu}\frac{a}{r}\exp\left(-\frac{r-a}{L_d}\right), \tag{4.22}$$

where $r$ is the distance of the source from the centre of the sphere.

Equation (4.22) demonstrates the way in which the interaction in an infinite medium of reflecting material may be expected to decrease with increasing distance between source and target, and this variation has been implied in putting forward the over-conservative $1/r$ law mentioned in Section 4.1.4. However, unless it can be demonstrated for the case in question that (4.22) will give conservative results this relation should not be used direct as it is well known that diffusion theory underestimates neutron densities near to sources but overestimates them at large distances. As a safe and probably useful upper limit to $p$, the probability of neutrons reaching the sphere without collision should be added to the value given by (4.22). An upper limit to this factor is $\exp[-(r-a)/L_t]$ multiplied by the fractional solid angle subtended by the sphere at a point distant $r$ from the centre.

Now consider the air-spaced reflected cores of case (ii). Let $\epsilon$ and $T$ denote the albedo and transmission of the shell to neutrons injected at the inner surface, and let $\epsilon'$ and $T'$ denote the corresponding quantities for neutrons injected at the outer surface, i.e. for one neutron incident upon the outer face $T'$ neutrons reach

the inner face and $\epsilon'$ return to the outer face. Then by arguments similar to those leading up to (4.20) it follows that the number of neutrons emerging from the reflected unit per neutron incident upon the outer surface is given by

$$M_{sr} = \varepsilon' + \frac{M_{sb}TT'}{1 - \varepsilon M_{sb}}. \tag{4.23}$$

Equation (4.23) is of little practical use as it stands but serves as a starting point for developing useful conservative approximations to $M_{sr}$. The form that any particular approximation takes may be dictated by circumstances but the following example is fairly typical.

If the central source multiplication for the system, $M_{cr}$, has been measured it will in fact be given by

$$M'_{cr} = \frac{M_{cb}T}{1 - \varepsilon M_{sb}} \div \frac{T}{1 - \varepsilon}, \tag{4.24}$$

the division by $T/(1 - \epsilon)$ being included because of the way in which experimental count rates are normalized, a dummy run being done with a pure scattering core (say graphite) in place of the fissile one. Then from (4.23, 24)

$$M_{sr} = \varepsilon' + \frac{TT'}{1 - \varepsilon}\frac{M_{sb}}{M_{cb}}M'_{cr} \tag{4.25}$$

It should be noted that various details such as changes in angular distribution of neutrons after transmission through the outer reflector have been glossed over in setting up (4.23). This will to some extent be rectified in (4.25) where the experimentally measured value of $M'_{cr}$ is employed. In any event such effects are probably swamped by the conservative approximations to be made later.

Equation (4.25) can be developed further as follows. Let $\omega_1$ and $\omega_2$ be the solid angles subtended at the reference point by the inner fissile core and the whole reflected unit, respectively. Assume that all neutrons incident upon the unit in solid angle $\omega_1$ reach the core (conservative). Neutrons from the source which travel inside the cone with solid angle $\omega_2$ but outside the cone with solid angle $\omega_1$ must have at least one collision before hitting the fissile core. Assume that one half of these neutrons hit the core (conservative), and that the remaining half all escape from the unit*. In this approximation

$$p\varepsilon' = \tfrac{1}{2} \cdot \frac{\omega_2 - \omega_1}{4\pi} \tag{4.26}$$

$$pT' = \frac{\omega_1}{4\pi} + \tfrac{1}{2}\frac{\omega_2 - \omega_1}{4\pi} = \tfrac{1}{2} \cdot \frac{\omega_2 + \omega_1}{4\pi}. \tag{4.27}$$

*In certain cases, say, with thin or low density reflectors, more than one half of these neutrons will escape. However in such cases the number which hit the core after scattering will have been overestimated, and it is these neutrons that carry most weight.

Further, $T + \epsilon \leqq 1$, i.e. $T/(1 - \epsilon) \leqq 1$. Then from (4.16, 19, 25, 26, 27) it is seen that the interaction parameter $q$ satisfies

$$q \leqq \frac{\omega_2}{4\pi} M'_{sr}$$

where

$$M'_{sr} = \tfrac{1}{2}\left[\left(1 - \frac{\omega_1}{\omega_2}\right) + \frac{2M_{cb} - 1}{M_{cb}}\left(1 + \frac{\omega_1}{\omega_2}\right) M'_{cr}\right] \tag{4.28}$$

Means of calculating more accurate values of $q$ will be considered in Section 4.4.

4.3.3 *Partially reflected systems.*—In Section 3.1.2 it was pointed out that the most dangerous situation can occur when two interacting units are reflected on the sides away from the other unit with no absorbing material between the units to prevent direct interaction. However, at ORGDP it is only anticipated that water can come between units and in this case they are immediately shielded from one another by the absorbing properties of the water. Obviously a more dangerous situation may arise if a purely scattering material (e.g. graphite or absorbent packing) is placed between partially reflected units for then neutron exchange by scattered paths may exceed the direct interaction obtained from air-spacing.

Such a situation is a compromise between cases (i) and (ii) given in Section 4.3.2 above and interaction parameters can obviously be deduced by a combination of the methods outlined there. In practical examples of partially reflected units that have occurred at AWRE some conservative fully reflected model of the actual unit has been adopted to obtain an estimate of the surface multiplication of each unit and if some interstitial scatterer is also present a method such as that outlined in Section 4.3.2 has been used to obtain the probability of neutrons reaching one unit from another.

## 4.4 *Calculation of Interaction Parameters*

Until quite recently little has been done at AWRE in the way of calculating values of $q$ to compare with experiment or to use in their own right, except for certain exceptional cases (e.g. the transit container problem mentioned in Section 4.2.2) when Monte Carlo techniques have been employed. The advantages of Monte Carlo methods using fast computers is (a) the accuracy of the results given good statistics, and (b) their applicability to non-spherical geometries. However, offset against (a) is the fact that tolerable accuracy requires a large amount of computer effort with the programmes at present available. In this account only the faster but more approximate methods that have been developed will be described.

Some work on the problem of calculating directly in two dimensions the interaction between two like spheres embedded in infinite scattering and absorbing material has also been carried out using simple diffusion methods. This will be published elsewhere in the near future.

4.4.1 *External spherical source programme; $S_n$-approximation.*—Very recently the familiar Carlson $S_n$-method (16) for solving the neutron transport equation has been adapted to computing the steady state neutron densities and (physical) fluxes maintained in a spherically symmetric subcritical system by neutrons emitted from a source in the form of a spherical surface. This source represents an aggregate of point sources all at the same radial position and, as the outputs due to non-multiplicative sources are additive, the total (symmetrical) output due to the surface source will be equal to the total (asymmetrical) output due to a point source of the same strength.

In the programme (EXSO) as it stands* the position of the source and the spectrum and angular distribution of the emitted neutrons are factors which can be controlled at will. A complete range of multiplication factors, albedos, and transmissions can therefore be covered. The basic limitation is the way in which scattering is treated. Allowance for anisotropy is only made in the diagonal terms of the (multigroup) scattering matrix so that strongly moderating materials cannot be coped with adequately. An estimate of the errors that do occur in such cases is being made.

Two examples of the results obtained using EXSO are given below.

4.4.2 *Multiplication factors for* 20 kg $U^{235}$ *spheres with and without graphite reflectors.*—The first system considered was a 20 kg sphere of 93 % $U^{235}$ at density 18·7 g/cm$^3$ and with a central cavity of radius 1·1 cm reflected by various thicknesses of graphite (density 1·6 g/cm$^3$). The source was placed in either of two positions, (1) at the surface of $U^{235}$, (2) at the outer surface of the graphite, and was isotropic with the conventional fission spectrum. $S_8$-approximation was used together with seven energy groups spanning the range 18 keV–10 MeV. Table 10 below gives the calculated values for the total number of neutrons crossing the boundaries (1) and (2) when one neutron is released at the source

TABLE 10

*Numbers of Neutrons per Unit Source Strength Crossing the Various Interfaces in the System*: 20 kg $U^{235}$ (93 %) *Sphere plus Graphite Reflector*

| Source position | (1) | | | | (2) | | | |
|---|---|---|---|---|---|---|---|---|
| Interface | (1) | | (2) | | (1) | | (2) | |
| Total Flux / Graphite thickness | In-going | Out-going | In-going | Out-going | In-going | Out-going | In-going | Out-going |
| 0 in. | 0·5 | 0·5+1·374 | | | | | | |
| 1 in. | 0·5+0·413 | 0·5+2·410 | 0 | 2·495 | 0·371 | 0·980 | 0·5 | 0·5+1·109 |
| 2 in. | 0·5+0·772 | 0·5+3·334 | 0 | 3·058 | 0·287 | 0·752 | 0·5 | 0·5+0·965 |

*Written for the IBM 709 by L. H. Underhill's group, Mathematical Physics Division, AWRE.

The direct contribution to the net flux from neutrons leaving the source, 0·5 for a unit isotropic source, is indicated in each case.

The values listed in Table 10 allow the various transmission, reflection, and multiplication factors used in Section 4.3 to be deduced as functions of the graphite thickness. These factors are listed in Table 11 below.

TABLE 11

*Surface Multiplications and Transmission and Reflection Factors for* 20 kg $U^{235}$ (93%) *Sphere in Graphite Reflector*

| Graphite thickness | 0 in. | 1 in. | 2 in. |
|---|---|---|---|
| $M_{sb}$ | 2·748 | 2·639 | 2·621 |
| $M_{sr}$ | 2·748 | 2·218 | 1·930 |
| $\epsilon$ | 0 | 0·142 | 0·201 |
| $T$ | 1 | 0·857 | 0·797 |
| $\epsilon'$ | 0 | 0·537 | 0·730 |
| $T'$ | 1 | 0·463 | 0·270 |

The variation in the surface multiplication $M_{sb}$, of the inner $U^{235}$ sphere alone as the graphite thickness increases is due to the slight difference in spectrum and angular distribution between neutrons reflected back from the graphite and neutrons emitted by the source. The "central source" multiplication calculated by the same means with the source surface on the outside of the 1·1 cm cavity is found to be 3·334.

If the values of $M_{sb}$, $\epsilon$, $T$, $\epsilon'$, and $T'$ given in Table 11 are inserted in equation (4.23) the values for $M_{sr}$ so obtained are in agreement with the calculated values.

4.4.3 *Calculation of the interaction between two undermoderated systems.*—In order to obtain clearance for storage prior to doing critical mass experiments it was desired to know the interaction between spheres*, each of volume 100 in.$^3$ composed of a $UO_2$/paraffin wax compact. The composition by atoms of the batch of material to be considered here was 4·5% $U^{235}$, 10·4% $U^{238}$, 18·2% C, 37·2% H, 29·7% O, with a bulk density of 6·42 g/cc, and so Henry's method of Section 3 is not applicable.

In applying the EXSO method to estimate the surface multiplication, $M_s$, of such spheres three difficulties present themselves:

(i) the limitations of the assumptions made concerning the scattering laws will preclude an accurate estimate of $M_s$;

(ii) the possible inaccuracies in the available nuclear data for $U^{238}$;

(iii) the angular distribution and spectrum to which the spheres are subjected in the array are unknown.

*The material was actually to be stored in blocks with dimensions 5 in × 5 in. × 4 in.

Regarding point (i) it was checked that the same calculating regime when applied to estimating critical sizes of $U^{235}/H_2O$ mixtures and solutions gave underestimates of these sizes over a wide range of $H/U^{235}$ ratios. It therefore seems certain that EXSO will overestimate the required surface multiplication.

Point (ii) was overcome conservatively by replacing the $U^{238}$, atom-for-atom, by $U^{235}$.

Considering (iii) use was made of the fact that the surface multiplication of a sphere is greatest when the neutrons are injected radially. In the calculations that were done the source was placed on the surface of the sphere and neutrons injected both radially and isotropically.

When situated in an array of identical spheres the spectrum that a sphere receives is exactly the same as the spectrum that it emits. The calculations were therefore carried out iteratively, first subjecting the sphere to a flux constant over velocity and noting the emergent spectrum and multiplication, feeding this spectrum back into the sphere as the source spectrum and noting the new emergent spectrum and multiplication, and so on until the emergent spectrum coincided with the source spectrum, or rather until the surface multiplication converged to the desired degree of accuracy.

The results of these calculations are given in Table 12 below.

TABLE 12

*Iterated Source Calculations on* $UO_2$*/Wax Compacts*

(100% $U^{235}$ enrichment)

| Radial source | | | Isotropic source | | |
|---|---|---|---|---|---|
| Source spectrum | Emergent spectrum | Surface multiplication | Source spectrum | Emergent spectrum | Surface multiplication |
| Uniform | (1) | 1·891 | Uniform | (1) | 1·747 |
| (1) | (2) | 1·642 | (1) | (2) | 1·616 |
| (2) | (3) | 1·634 | (2) | (3) | 1·566 |
| (3) | (4) | 1·634 | (3) | (4) | 1·546 |

It is seen that, for this system, $M_s$ is not sensitive to variations in the angular distribution and spectrum of the source. The change with angular distribution is small since the diameter of the sphere represents several neutron mean free paths for the fast neutrons.

Table 13 opposite shows the change in source spectrum with successive iterations for the radial source case. The quantities listed are the neutron fluxes per unit total source strength (i.e., per neutron per unit area per second released by the surface source) in each of the eight velocity groups employed.

Thus although the converged flux profile is significantly different from the uniform one assumed for the first iteration the overall multiplication is only changed by 10%.

TABLE 13

*Change of Neutron Flux Profile with Successive Iterations*

| Neutron group \ Iteration | Uniform | (1) | (2) | (3) | (4) |
|---|---|---|---|---|---|
| Group 1<br>11 MeV – 1·6 MeV | 0·125 | 0·32654 | 0·29545 | 0·28851 | 0·28772 |
| Group 2<br>1·6 MeV – 0.78 MeV | 0·125 | 0·19214 | 0·17765 | 0·17272 | 0·17182 |
| Group 3<br>0·78 MeV – 0·18 MeV | 0·125 | 0·21733 | 0·22655 | 0·22322 | 0·22167 |
| Group 4<br>0·18 MeV – 10 keV | 0·125 | 0·19290 | 0·24485 | 0·25796 | 0·26022 |
| Group 5<br>10 keV – 0·26 keV | 0·125 | 0·04555 | 0·04601 | 0·04859 | 0·04940 |
| Group 6<br>0·26 keV – 15 eV | 0·125 | 0·01304 | 0·00713 | 0·00717 | 0·00732 |
| Group 7<br>15 eV – 0·4 eV | 0·125 | 0·00995 | 0·00225 | 0·00180 | 0·00181 |
| Group 8*<br>0·4 eV – 0 | 0·125 | 0·00256 | 0·00012 | 0·00004 | 0·00004 |
| Surface multiplication | | 1·8194 | 1·6424 | 1·6344 | 1·6343 |

*Thermal group with cross sections averaged over Maxwellian distribution.

As an example of applying these results for $M_s$ to the clearance of an array of $UO_2$/wax units consider an air-spaced array of 100 units. From equation (4.7) these units can be arranged in any geometry provided that no two units are closer together than would make $q$ exceed the value 1/99. Taking $M_s = 2$ as an absolute upper limit this requires that no unit can subtend a solid angle fraction of more than 1/198 at any other unit, and this in turn corresponds to a minimum edge-to-edge separation of 20 in.

If in practice it were proposed to place this array, say, on a concrete floor, then some extra allowance must be made for the reflecting properties of the floor.

## 5. COMPARISON OF THE ORGDP METHOD WITH THE INTERACTION PARAMETER METHOD

From what has been said in Sections 3 and 4 it is clear that in the method adopted by Henry *et al.* the solid angle fraction and the $k$-factor correspond, respectively, to the general transmission probability $p$ (which for air-spaced arrays is equal to the solid angle fraction) and the surface multiplication, $M_s$. However, it could be claimed that the factor $M_s$ has more immediate physical significance than the theoretical factor $k$ in the sense that a large value of $M_s$ conveys more quickly the potential danger of an object in the array than does the nearness to unity of the $k$-value for that object. For this reason and also so that $q$-values for thermal systems may be deduced from $k$-values it is desirable to have available some relationship, preferably simple, between $M_s$ and $k$.

### 5.1 *Upper Limits to $M_s$ and $q$ in terms of the k-factor*

From the definition of $k$, one neutron released in the eigen-distribution of the system produces $k$ neutrons in the next generation. These $k$ neutrons produce $k^2$ neutrons in the next generation, and so on, the eventual multiplication being

$$1 + k + k^2 + \ldots = \frac{1}{1-k},$$

provided $k$ is less than unity.

In the notation of Section 3.6 a fraction $(1-U_f)$ of the fast neutrons produced per generation escape from the system. The fraction $U_f$ remain in the system to be slowed down and produce $U_fU_t$ thermal captures and $U_f(1 - U_t)$ thermal escapes. The total number of neutrons escaping from the system per generation is therefore

$$(1 - U_f) + U_f(1 - U_t) = 1 - U_fU_t = 1 - \frac{k}{\eta f}$$

The number of neutrons which emerge from the sphere per neutron born in the eigen-distribution is therefore given by

$$\bar{M} = \frac{1 - (k/\eta f)}{1 - k} \tag{5.1}$$

As pointed out in Section 4.3.1 the number of neutrons which emerge from a system per neutron released isotropically at some point within the system will, in most cases of interest, vary monotonically between the values $M_c$, for neutrons released at the centre, to $(M_s + 1)/2$ for neutrons injected at the surface. Under these circumstances the weighted mean multiplication $\bar{M}$ will satisfy

$$M_c \geq \bar{M} \geq \tfrac{1}{2}(M_s + 1), \tag{5.2}$$

and so from (5.1)

$$M_s \leqq \frac{1 + [1 - (2/\eta f)]k}{1 - k}. \tag{5.3}$$

Over the range of applicability of the Henry method the factor $\eta f$ is virtually equal to 2 so that the factor $1/(1-k)$ of the Henry method bears favourable comparison with the $M_s$ of the interaction parameter method, although for general calculation purposes the factor $[1-(2/\eta f)]$ in (5.3) should not be discarded.

For the air-spaced systems considered by Henry the transmission probability, $p$, is equal to the solid angle fraction $\Omega/4\pi$, and so from (5.3)

$$q \leqq \frac{1+[1-(2/\eta f)]k}{1-k} \cdot \frac{\Omega}{4\pi} \tag{5.4}$$

Equation (5.4) can be used to deduce conservative values of $q$ for the types of systems covered by Henry's recipe for evaluating $k$, and the rules governing the use of $q$-values as set out in Section 4.1 apply.

5.2 *Numerical Examples and Checks*

The relation (5.4) might be criticized on the grounds that (5.2) has only been demonstrated for metal spheres and that therefore (5.3) and (5.4) might give erroneous results when applied to the types of thermal system considered by Henry and Callihan. To check this (5.4) will be used to calculate some lattice spacings that correspond to critical experiments carried out by Callihan (see Fig. 4).

Consider two identical containers containing fissile solution with $H/U^{235}$ ratio equal to 93·75. This value is chosen so as to make $\eta f$ equal to 2 and so (5.4) reduces to

$$q \leqq \frac{1}{1-k} \cdot \frac{\Omega}{4\pi}$$

In the critical state $q$ is equal to unity (cf. equation 4.5) so that the solid angle fraction satisfies

$$\frac{\Omega}{4\pi} \geqq 1-k$$

The straight line in Fig. 4 where the solid angle fraction is equal to $(1-k)$, shown as the broken curve $C$, should therefore represent a safety line for this $H/U^{235}$ ratio, with critical systems all lying above the line and subcritical systems below. As it is seen all the experimental points quoted lie above curve $C$ with the exception of one point which refers to two 20 in. diameter cylinders of height 5·83 in. containing solution with $H/U^{235} = 169$ and this lies fractionally below the line. However, the $k$-values for these cylinders (0·964) exceed the safe limit (0·9) imposed by ORGDP on bare cylinders.

As a further check example consider an hexagonal array of six cylinders, 8 in. diameter and containing $UO_2F_2$ solution with $H/U^{235}$ ratio of 44·3, placed symmetrically about a central unit. Let the neutron output of the outer cylinders be $F$ and the output of the central cylinder be $G$. Let $d$ be the spacing of adjacent cylinders.

Then as the central cylinder "sees" all six surrounding cylinders

$$G = 6q(d)F. \tag{5.5}$$

Each outer cylinder "sees" two other outer cylinders at distance $d$, two more at distance $\sqrt{(3)}d$, and the central cylinder at distance $d$, so that

$$F = 2[q(d) + q(\sqrt{(3)}d)]F + q(d)G \tag{5.6}$$

For simplicity assume that $q(\sqrt{(3)}d) = q(d)/3$, using the inverse square law for air-spaced lattices; for near spacings this is approximate but in this case will give a conservative result. Then equations (5.5, 6) provide a quadratic equation for $q(d)$ with the solution $q(d) = 0{\cdot}2426$ in the critical array. Hence from (5.4) with the appropriate value of $\eta f$, the critical solid angle fraction for two adjacent cylinders satisfies.

$$\frac{\Omega}{4\pi} \geqq \frac{\Omega_c}{4\pi} \equiv 0{\cdot}2426\frac{1-k}{1+0{\cdot}0227k} \tag{5.7}$$

Table 14 below compares the values of $\Omega_c/4\pi$ with the solid angle fractions measured in critical experiments on such arrays[3].

TABLE 14

*Comparison of $\Omega_c/4\pi$ with Measured Critical Solid Angle Fractions*

| Cylinder height (in.) | $k$ for single cylinder | Measured critical solid angle fraction | $\Omega_c/4\pi$ (5·7) |
|---|---|---|---|
| 7·2 | 0·7448 | 0·159 | 0·0609 |
| 10·1 | 0·8343 | 0·102 | 0·0395 |
| 13·1 | 0·8836 | 0·075 | 0·0277 |
| 16·4 | 0·9146 | 0·062 | 0·0203 |
| 22·0 | 0·9425 | 0·043 | 0·0137 |

As before it is seen that the use of (5.4) with the equality sign gives conservative estimates for critical solid angle fractions, thus indicating that (5.4) does indeed provide an upper limit to $q$.

It is worthy of mention that if the relation (4.11) is used in the last example in place of the accurate equations (5.5, 6), the numerical coefficient in (5.7) is reduced to the value $\frac{1}{6}$ and the corresponding results for $\Omega_c/4\pi$ are even more conservative.

### 5.3 *On the Interaction Between Pairs of Dissimilar Containers*

Relations (4.4) and (5.4) enable a check to be made upon the validity of Henry's empirical rule for dissimilar containers mentioned in Section 3.4.

Consider two dissimilar containers with $k$-values $k_1$ and $k_2$, and assume in the first instance that the separation distances involved are large enough for the solid angle fractions to be represented by the cross-sectional areas of the containers, $A_1$ and $A_2$, say, divided by $4\pi$ times the square of the separation distance. Then from relations (4.4) and (5.4) the two dissimilar containers are safe at a mutual separation distance $d_{12}$ satisfying

$$\frac{1+[1-(2/\eta_1 f_1)]k_1}{1-k_1}\cdot\frac{A_1}{4\pi d_{12}^2}\cdot\frac{1+[1-(2/\eta_2 f_2)]k_2}{1-k_2}\cdot\frac{A_2}{4\pi d_{12}^2}<1 \qquad (5.8)$$

For pairs of like containers of types 1 and 2 to separately form subcritical systems their respective $q$-values must be less than unity, i.e. their respective separations $d_1$ and $d_2$ must satisfy

$$\frac{1+[1-(2/\eta_i f_i)]k_i}{1-k_i}\cdot\frac{A_i}{4\pi d_i^2}<1, \quad i=1,2. \qquad (5.9)$$

Hence from (5.8, 9) the separation between dissimilar containers must satisfy

$$d_{12}^2 > d_1 d_2, \qquad (5.10)$$

i.e. that $d_{12}$ must exceed the geometric mean of $d_1$ and $d_2$ whereas Henry's empirical rule requires that $d_{12}$ is equal to (or exceeds) the arithmetic mean of $d_1$ and $d_2$. It is well known that the arithmetic mean of two quantities always exceeds the corresponding geometric mean and so Henry's rule is in fact conservative as the experiments that he mentions appear to demonstrate.

The argument used above can immediately be generalized to eliminate the assumption regarding the dependence of solid angle fraction upon separation distance. This assumption was only introduced to give a more immediate comparison with Henry's rule as it stands.

Let $w_{ij}$ $(i,j=1,2,)$ be the solid angle subtended by a container of type $i$ at the position of another container of type $j$, $i=j$ implying like containers. Then if $w_{11}$, $w_{22}$ represent *safe* values for pairs of like containers, pairs of unlike containers will be safe, generalizing (5.10), provided

$$w_{12}w_{21} < w_{11}w_{22}. \qquad (5.11)$$

These considerations of course apply only to air-spaced arrays.

## 6. GENERAL CONCLUSIONS

For assessing, criticality-wise, arrays of metal spheres, bare or reflected, use should be made of either the Los Alamos experimental data (Section 2) or of the interaction parameter method via measured or calculated values of surface multiplications. The former is the more direct method and should be used where possible, although further investigations along the lines suggested by Ketzlach are necessary before any rules of thumb can be laid down. The latter method

is the more tedious to apply but has the advantage that once the necessary interaction parameters have been estimated the degree of criticality in any geometry can be assessed. The use of Fig. 1 for very large arrays (say, greater than 100 units) is not advised until further corroboratory evidence is available.

For arrays consisting of well-moderated units of general assay the ORGDP method is applicable provided that the $k$-factors for the individual units are evaluated as prescribed in Section 3. In view of the fact that little is stated about the experiments performed with dissimilar interacting units it would be advisible to treat with caution the simple rule given in Section 3.4 when the $k$-values for the individual containers are widely different, pending a check using relation (5.11). In fact it would be of great utility to have relation (5.11) checked experimentally over as wide a range of dissimilar containers as possible, for in the case where reasonably large separations are involved the geometric mean of the safe separations for the pair of like containers involved can be considerably less than the arithmetric mean, and consequently a large amount of storage space may be saved.

The interaction parameter method has general application to arrays for which reasonably accurate or conservative estimates of the parameters can be found. The chief drawbacks which arise in practice are:

(i) the measurement of $q$-values for systems completely immersed in reflecting material, and in particular water and other hydrogenous materials;

(ii) the difficulty of obtaining accurate calculations in these cases, although these are to some extent being overcome by Monte Carlo methods on fast computers;

(iii) the translation of measurements or calculations made with point sources into data for two-body systems when separation distances are small; usually some conservative device has to be adopted.

Cases of awkward geometry can give rise to difficulties but no more so than those presented by other methods.

As a general rule the interaction parameter method should be used when it is either not possible or not convenient to use the other methods described.

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## REFERENCES

1. L. GELLER, Carbide and Carbon Chemicals Company, Report KS–317, Sept. 1952.
2. H. K. CLARK, A.E.C. Research and Development Report DP–312, November 1958.
3. H. F. HENRY, J. R. KNIGHT and C. W. NEWLON, Union Carbide Nuclear Company Report K–1309, November 1956.
4. R. T. ACKROYD and J. D. MCCULLEN, Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva 1958, Vol. 12, P/16.

5. E. C. MALLARY, H. C. PAXTON and R. H. WHITE, LA–1875 (Deleted), April 1958.
6. H. C. PAXTON, LA–1958.
7. N. KETZLACH, Nuclear Physics Research Quarterly Report (Hanford Laboratories), HW–56919, July 1958.
8. C. V. LARRICK, Knolls Atomic Power Laboratory, Report KAPL–M–CVL–4, January 1956.
9. H. F. HENRY, C. E. NEWLON and J. R. KNIGHT, Union Carbide Nuclear Company, Report K–1317, December 1956.
10. "Studies in Nuclear Safety" edited by H. F. Henry. Union Carbide Nuclear Company, Report K–1380, August 1958.
11. A. D. CALLIHAN, D. F. CRONIN, J. K. FOX, R. L. MACKLIN and J. W. MARFITT, Carbide and Carbon Chemicals Company, Report K–406, November 1949.
12. A. D. CALLIHAN, Unpublished data.
13. J. K. FOX and L. W. GILLEY, Oak Ridge National Laboratory, Report CF–56–7–148, July 1956.
14. C. M. NICHOLLS, E. R. WOODCOCK and A. H. GILLIESON, "Criticality", a chapter in *The Processing of Reactor Fuels*, Editor J. Flagg. To be published. Academic Press.
15. A. J. MAYNE, *J. Nucl. Energ.*, **2**, 77 (1955).
16. B. CARLSON, LA–1891.