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AN EMPIRICAL STUDY OF SOME CRITICAL MASS DATA

By

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Y-12 PLANT
CARBIDE AND CARBON CHEMICALS CORPORATION
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AN EMPIRICAL STUDY OF SOME CRITICAL MASS DATA

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ABSTRACT

A simple empirical equation has been found which relates the critical height of a water-enclosed stainless steel reactor at a given moderation to its diameter. Certain empirical constants appearing in the relation have a simple physical interpretation which succeeds to a limited extent in bridging the gap between the experimental results with finite cylindrical reactors and Grueling's[1] theoretical treatment which is limited to infinite cylinders or slabs. The report also discusses certain comparisons between theoretical and experimental results.

Oak Ridge, Tennessee
December 6, 1949

1. Grueling, E., "Theory of Water-Tamped Water Boiler", LA-399, September 27, 1945.

AN EMPIRICAL STUDY OF SOME CRITICAL MASS DATA

INTRODUCTION

The results of a series of critical mass experiments performed in cylindrical geometry and utilizing aqueous solutions of uranyl fluoride of 93.4% U_{235} isotopic assay have been reported in "Critical Mass Studies, Part III".[2] The application of curve fitting methods to certain of the empirical data there presented has a three-fold purpose.

1. To express the results of Part III in a more compact form and to make interpolation of data a more convenient process.
2. To allow estimation of critical conditions in a region beyond the range of the experimental data.
3. To provide a means for estimating the critical diameters of cylinders of infinite length and the critical thickness of slabs of infinite surface area for comparison with the critical dimensions obtained by the integral theory of Grueling.[3]

In addition this particular approach may, in time, provide simpler and more direct mathematical treatments than currently exist in regard to certain basic problems.

PROCEDURE AND RESULTS

The curves of Figure 1 were obtained by plotting the experimentally determined critical height of a cylinder as a function of its diameter for

2. Beck, C. D., A. D. Callihan, J. W. Morfitt, R. L. Murray, "Critical Mass Studies, Part III", Carbide and Carbon Chemicals Corporation, K-343, April 19, 1949. (This report will hereafter be referred as Part III.)
3. Grueling, E., "Theory of Water-Tamped Water Boiler", LA-399, September 27, 1945.

various moderations in the manner given in Part III. Simple physical

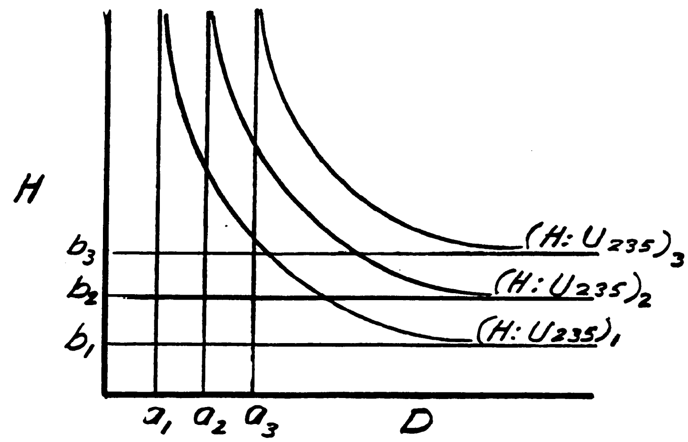


Figure 1.

considerations, confirmed by experiment, show that each curve exhibits two asymptotes which are readily interpreted. The vertical asymptote, D = constant, represents the diameter of a just critical infinite cylinder at the given moderation. Similarly, the horizontal asymptote, H = constant, represents the height of a just critical infinite slab. There is a pair of asymptotes for each curve corresponding to a different $H:U_{235}$ ratio.

These data were submitted to various curve fitting techniques and the most satisfactory fit was found to be a hyperbola of the form $(D-a)(H-b) = c$ where a and b are the respective asymptotes interpreted above. The method by which the empirical equations were obtained from the distribution of experimental points follows a procedure described by Running.[4] Its application to this problem is worked out in detail in the Appendix. By the method there described values of a, b and c can be readily obtained for each $H:U_{235}$ ratio, and are listed in Table I.

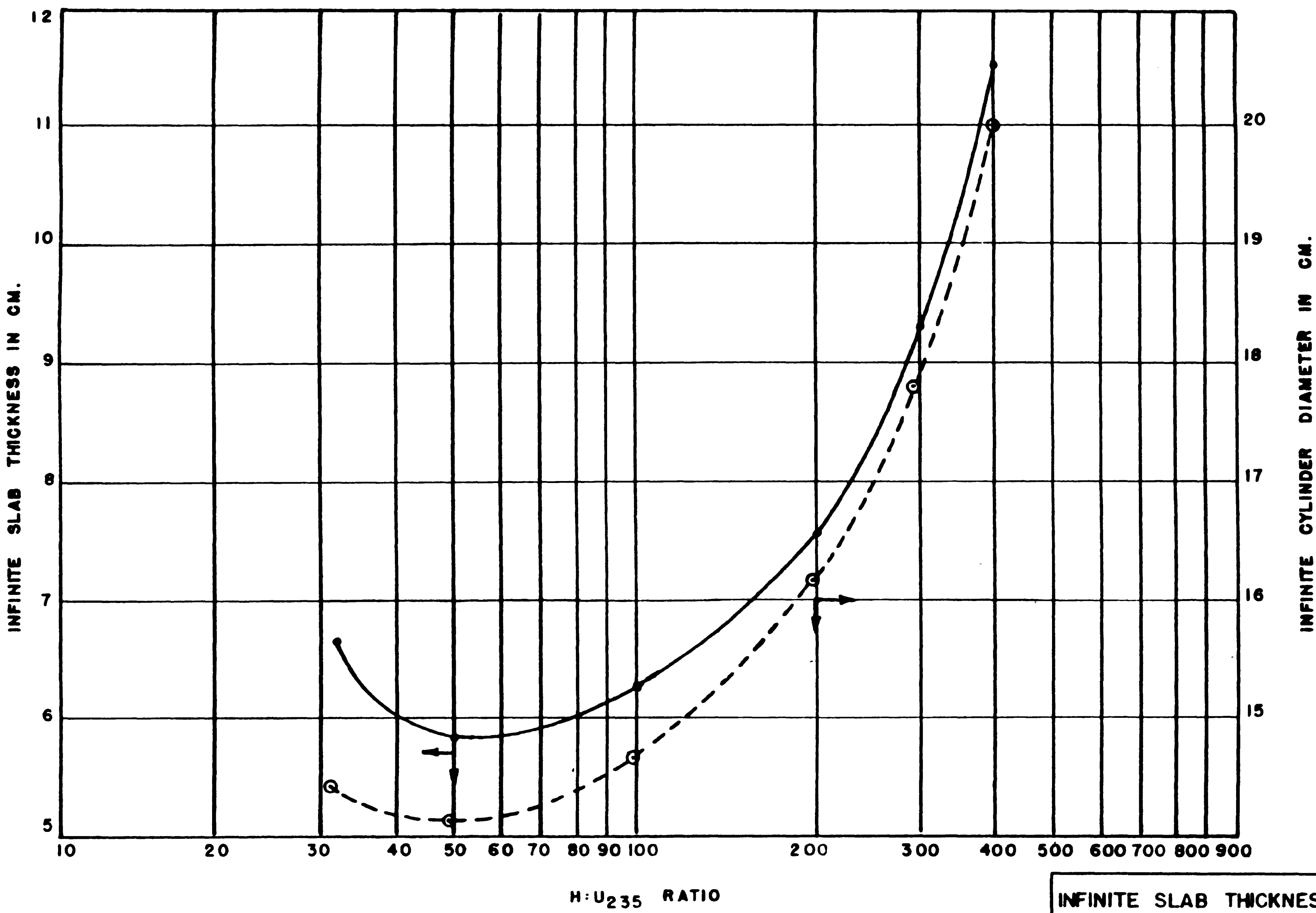
4. Running, Theodore R., "Empirical Formulas", John Wiley & Sons, New York, New York.

TABLE I

Limiting Dimensions of Infinite Slab and Cylinder at Various $H:U_{235}$ Ratios

$H:U_{235}$	Height of Infinite Slab(cm) b	Diameter of Infinite Cylinder(cm) a	c
31.6	6.42	14.22	99.0
50.0	5.85	14.10	102.6
100.0	6.24	14.58	104.1
200.0	7.60	16.10	104.9
300.0	9.25	17.88	108.4
400.0	11.54	20.02	104.1
		Average	104.0

Figure 2 is a plot of the data of Table I, i.e., the thickness of a just critical infinite slab (b) and the diameter of a just critical infinite cylinder (a) as a function of $H:U_{235}$ ratio. Having obtained the necessary constants, one can then use the formula $(D-a)(H-b) = c$ to calculate the critical height as a function of reactor diameter and moderation. The values so obtained when compared with the experimental values provide a test of the fit of the empirical equation. Such a comparison is made in columns 3 and 4 in Table III below, which gives the calculated and experimentally obtained values of the critical height for various size reactors under various conditions of moderation. It will be noted that the agreement is well within the reproducibility of the experimental data, and that the value of c varies only slightly with $H:U_{235}$ ratio. There is not sufficient evidence to indicate whether the variation in c is random or is significant. The magnitude of the inherent inaccuracies in the experimental data preclude the possibility of further refining the empirical technique. If c is allowed to vary in the manner shown in Table I its



INFINITE SLAB THICKNESS AS A FUNCTION OF MODERATION

INFINITE CYLINDER DIAMETER AS A FUNCTION OF MODERATION

FIGURE 2

INFINITE SLAB THICKNESS & CYLINDER DIAMETER AS A FUNCTION OF MODERATION

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calculation for the intermediate moderations becomes inconvenient. An average value of c for all moderations was used to simplify computation. Using this average value of c, together with the original experimental data, values of a and b were recalculated to give the best fit under these conditions. The new values of a and b were within about two millimeters of the originally obtained values as shown below in Table II. The assumption that c = constant for all $H:U_{235}$ ratios implies that all curves shown in Figure 1 have same shape and differ only in the location of the asymptotes.

TABLE II

Limiting Dimensions of Infinite Slabs and Cylinder Using Average Value of c = 104

$H:U_{235}$ Ratio	Height of Infinite Slab(cm) b	Diameter of Infinite Cylinder(cm) a
30	6.70	14.1
50	6.12	14.0
100	6.45	14.6
150	7.00	15.4
200	7.70	16.2
250	8.55	17.0
300	9.47	17.9
350	10.50	18.9
400	11.65	20.1

It is interesting to note that the minimum diameter of a just critical infinite cylinder is slightly over 5.5 inches.

Using the data of Table II, the critical heights were recalculated for these somewhat simplified conditions and are reproduced in column 5 of Table III below:

TABLE III

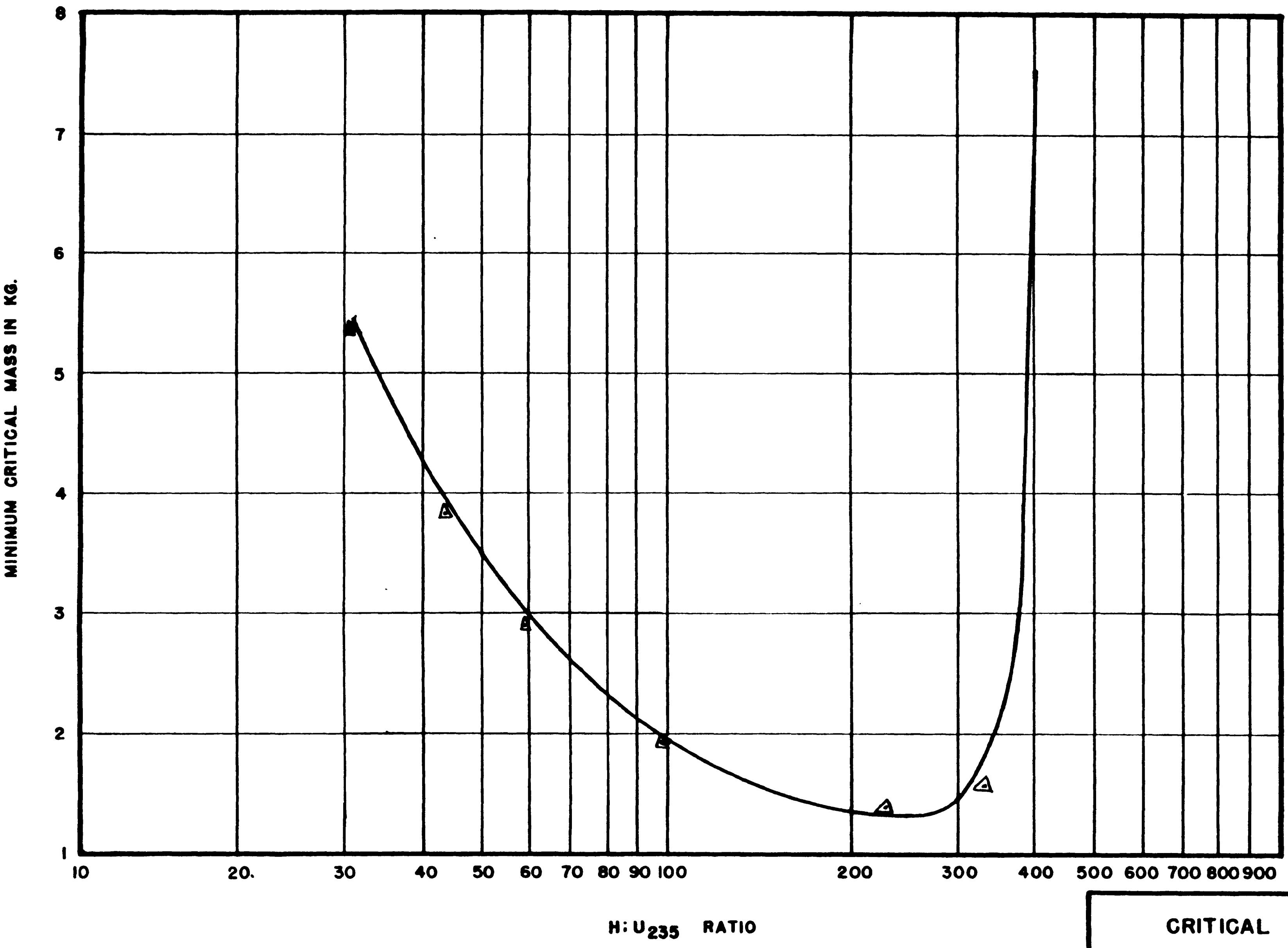
Comparison of Calculated and Experimental Values of Critical Height

Reactor Diameter (inches)	$H:U_{235}$ Ratio	Critical Height: From Part III (cm)	Critical Height: Using Values of c from Table I (cm)	Critical Height: Using Average Value (c = 104) (cm)
15	50.0	10.1	10.1	10.0
	100	10.6	10.6	10.4
	200	12.5	12.4	12.5
	300	14.7	14.6	14.6
	400	17.3	17.3	17.4
12	30.0	12.8	12.8	13.1
	50.0	12.6	12.1	12.4
	100	13.0	12.7	13.0
	200	15.1	14.9	15.0
	300	17.7	17.8	17.8
	400	21.5	21.4	21.7
10	30.0	15.4	15.6	15.9
	50.0	15.0	14.9	15.3
	100	15.6	15.8	16.1
	200	18.8	19.0	19.1
	300	23.7	23.7	23.4
	400	---	30.7	31.2
9	30.0	18.3	18.2	18.6
	50.0	17.8	17.6	17.9
	100.	18.9	18.7	19.1
	200	23.4	23.4	23.5
	300	31.2	31.0	30.5
	400	---	47.8	48.9
8	30.0	23.0	22.9	23.5
	50.0	22.2	22.3	22.6
	100	24.4	24.2	24.8
	200	32.8	32.9	33.3
	300	53.6	53.7	52.6
7	30.0	34.2	34.1	35.2
	50.0	32.4	33.7	33.8
	100	37.8	38.5	37.8
	200	---	73.2	76.0
6½	30.0	48.6	49.4	50.0
	50.0	47.1	48.4	47.9
	100	62.6	59.8	62.6

Comparing the simplified calculated values in column 5 with the experimental values of column 3, it is found that the average deviation from the experimental values is 1.4% with a maximum deviation of 5%, well within the expected experimental error.

The critical dimensions having been obtained, the mass may be calculated in the usual way. Figure 3 is a plot of the critical mass so obtained as a function of the $H:U_{235}$ ratio for a reactor eight inches in diameter, and illustrates that the general shape of such a curve agrees with its experimental counterpart in Part III and in addition provides a semi-quantitative idea of the steepness of the curve beyond the minimum. The experimental points have been plotted for comparison.

The same method may be applied to calculate the minimum critical mass in cylindrical geometry at a given moderation irrespective of cylinder dimensions. For any group of cylinders having the same volume the minimum surface area occurs for that cylinder where $H = D$. If it is assumed that the leakage in a critical water-enclosed cylinder is proportional to the surface area, the equilateral cylinder will have the smallest leakage and hence will need the smallest mass to make it critical. Following this reasoning the minimum critical masses was calculated as a function of the moderation on the assumption that $H = D$ for such cylinders. Elementary theory predicts $H = .92D$ for a reactor without an external water reflector. The ratio of H:D would be expected to be still closer to unity for the reflected case. Since the expected deviation of H/D from unit is of the order of the experimental error, the assumption that $H = D$ for minimum critical mass is satisfactory.



REACTOR DIAMETER 8."
STAINLESS STEEL REACTOR
WATER REFLECTOR
Δ-EXPERIMENTAL POINTS

FIGURE 3

CRITICAL MASS
AS A
FUNCTION OF MODERATION

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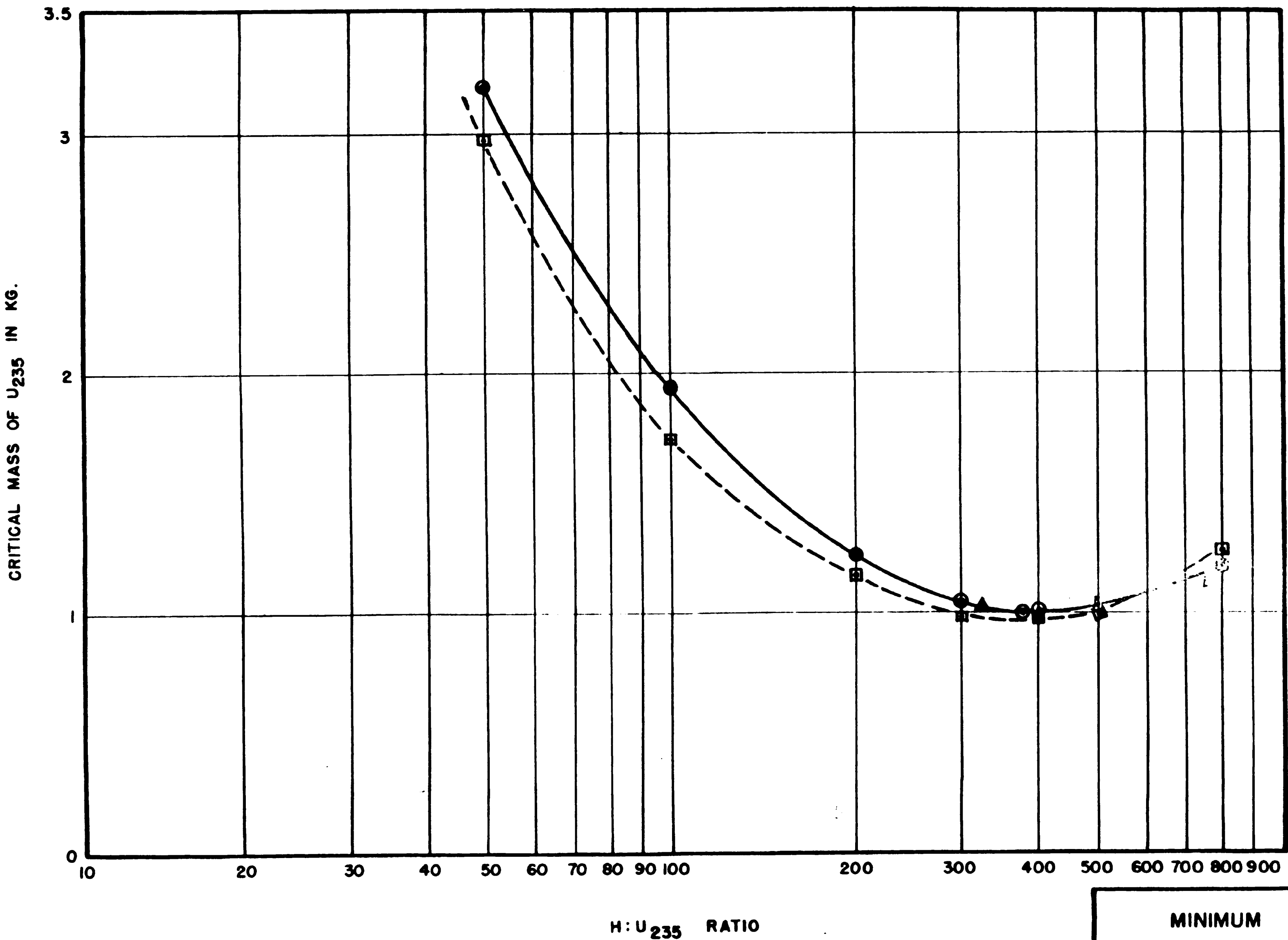
The critical mass was therefore calculated at various moderations for the case H = D and the results plotted in Figure 4. A curve showing the theoretically calculated diameter of the critical water-reflected sphere as a function of moderation has been included for comparison. A sample calculation is given in the appendix. The broad minimum which extends from $H:U_{235}$ of 200 to $H:U_{235}$ of 800 is partially explained by reference to Figure 5. The latter consists of two relationships, (1) minimum critical volume and (2) grams of U_{235} per unit volumn, both plotted as functions of the moderation. The critical mass curve is obtained by multiplying the ordinates of the two curves together for each value of the moderation. The fact that the critical mass is the product of an increasing function and a decreasing one accounts for the flatness of the minimum.

COMPARISON WITH THEORY

An added advantage in having obtained the limiting dimensions of the infinite slab and cylinder is that they can be compared with values obtained by theoretical methods, as for example, the integral method of Grueling.[5] The results reported there for infinite cylinders and slabs have been recalculated using the appropriate H and G functions with more recently reported values of the constants[6,7] (tabulated in Table VII),

5. Grueling, E., "Theory of Water-Tamped Water Boiler", LA-399, September 27, 1945.
6. Way, K., "Description of an Average Fission", MonP-192, December 10, 1946.
7. Way, K., G. Haines, "Thermal Neutron Cross Sections for Elements and Isotopes H-Bi", AECD-2138, October 11, 1948.



⊙ - EMPIRICAL VALUES FOR EQUILATERAL CYLINDRICAL REACTORS

⊡ - THEORETICAL VALUES FOR SPHERICAL REACTORS (LA 399) 9-27-45

▲ - EXPERIMENTAL VALUES FOR APPROXIMATELY EQUILATERAL CYLINDRICAL REACTORS

FIGURE 4

MINIMUM MASS AS A FUNCTION OF MODERATION	
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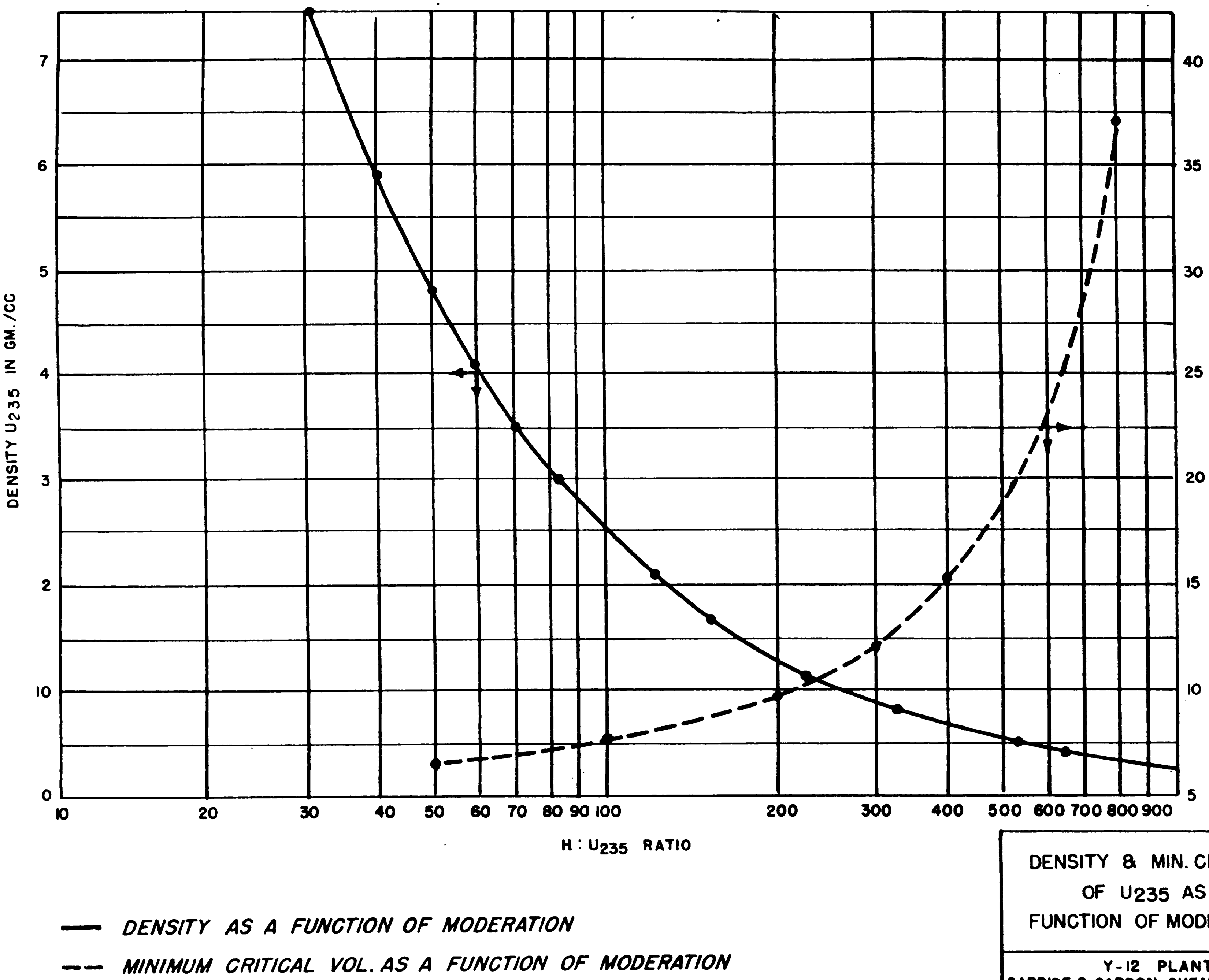

FIGURE 5

DENSITY & MIN. CRIT. VOL. OF U_{235} AS A FUNCTION OF MODERATION	
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and neglecting the effect of displacement on the core functions. The limiting dimensions obtained through the use of LA-399 can then be compared with the results obtained by the empirical method described here. The comparison is tabulated in Table IV and a graphical comparison is given in Figures 6 and 7.

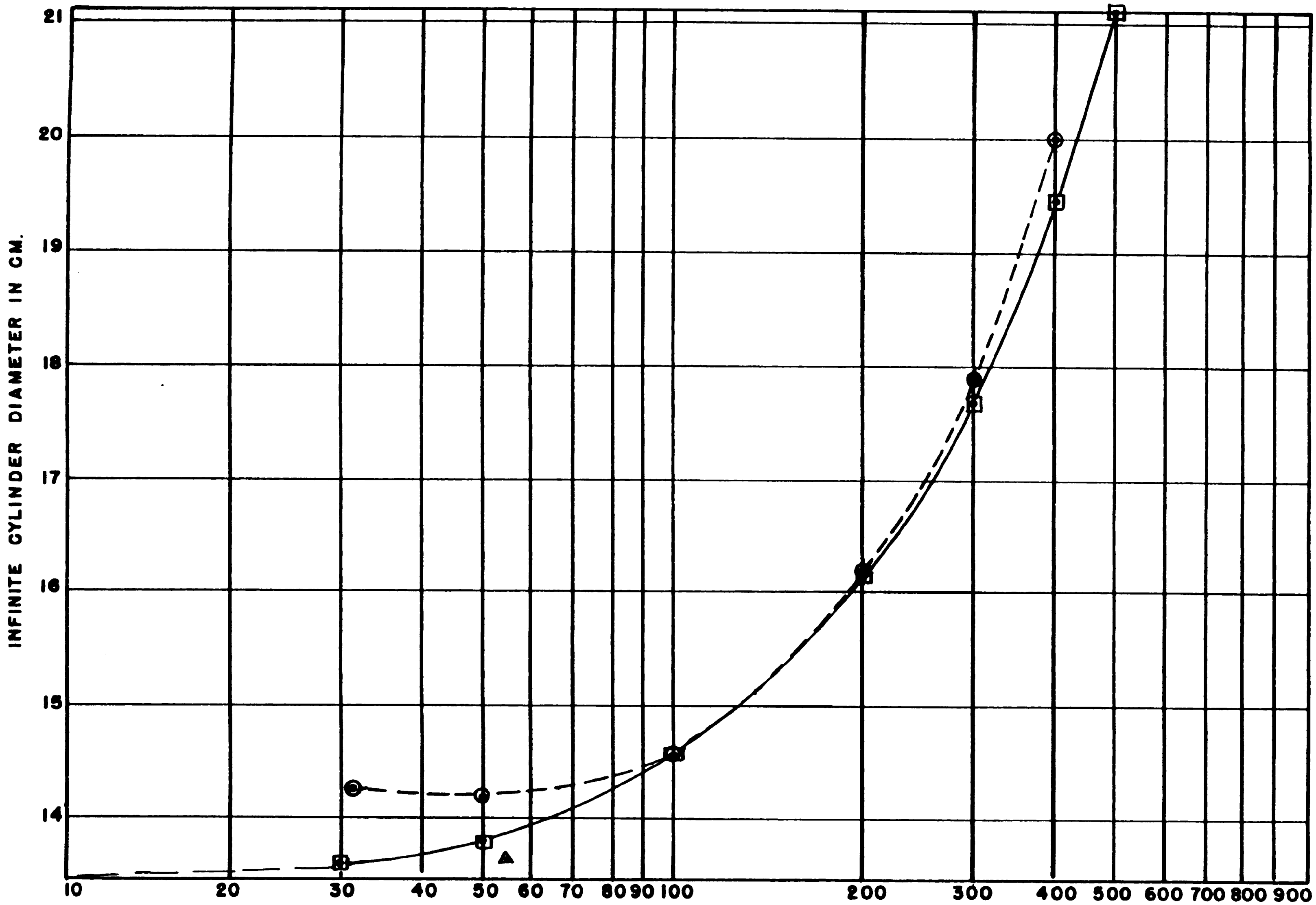
TABLE IV

Comparison of Empirically Determined Limiting Slab and Cylinder Dimensions with those Obtained by Methods of LA-399

	Slab Height		Cylinder Diameter		
$H:U_{235}$	Grueling (cm)	Empirical Stainless Steel(cm)	Grueling (cm)	Empirical Stainless Steel(cm)	Empirical Constants
30	4.66	6.70	13.61	14.1	101
31.6		6.42		14.22	99
50	4.95	5.85	13.89	14.10	102.6
100	5.51	6.24	14.58	14.58	104.1
200	6.75	7.60	16.10	16.18	104.9
300	7.85	9.25	17.60	17.88	108.4
400	9.02	11.54	19.48	20.02	104.1
500	10.08		21.11		104

When the effect of displacement on the H and G functions is considered, the slab and cylinder dimensions are increased slightly. Three such points have been plotted on the infinite slab graph to illustrate the amount of increase. The effect of introducing the displacement into the theoretical computation in the amount required by Grueling[8] is still not sufficient to predict an increase in the critical dimensions below an $H:U_{235}$ of 50 found empirically. It may therefore be that displacement

8. Grueling, E., "Theory of Water-Tamped Water Boiler", LA-399, September 27, 1945.



◉ - *EMPIRICAL VALUES*

⊡ - *THEORETICAL VALUES CALCULATED FROM (LA399) 9-27-45*

▲ - *EMPIRICAL VALUE FOR ALUMINUM CYLINDRICAL REACTOR*

FIGURE 6

INFINITE CYLINDER DIAMETER AS A FUNCTION OF MODERATION	
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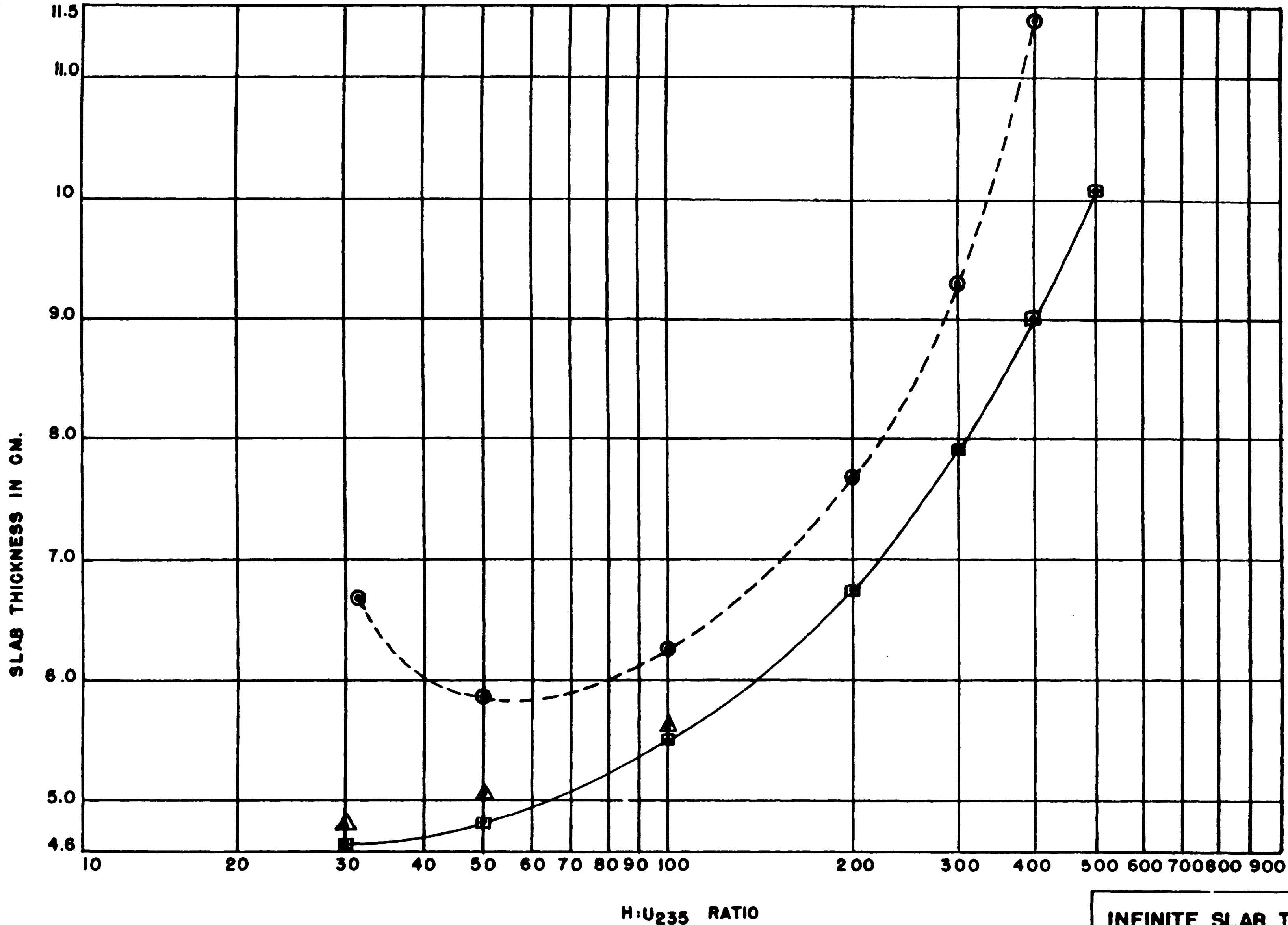

⊙: *EMPIRICAL VALUE OF SLAB*

⊡: *THEORETICAL VALUES CALCULATED FOR (LA399) 9-27-45*

△: *CORRECTION ON THEORETICAL VALUES FOR EFFECT OF DISPLACEMENT ON H(C) AND G(C) FUNCTION*

FIGURE 7

INFINITE SLAB THICKNESS AS A FUNCTION OF MODERATION	
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has a much larger effect than has been predicted.

It should be pointed out that the empirical treatment is based on data in stainless steel reactors while the theoretical treatment neglects wall effects. Sufficient data were available for an aluminum cylinder at a $H:U_{235}$ ratio of 55 (observed minimum critical height at this moderation) to calculate empirically the infinite slab and infinite cylinder values. These values along with the Grueling values appear in tabular form below:

TABLE V

Infinite Slabs and Cylinders @ $H:U_{235}$ = 55 for Aluminum Cylinders

	Theoretical	Empirically Derived Values
Infinite Slab, Height (cm)	5.0	4.3
Infinite Cylinder, Diameter (cm)	13.95	13.8

The values for the cylinder diameters fall well within the experimental error for both aluminum and stainless reactors, but the deviation of the values for the slabs is significant. It is not known whether the discrepancy is due to inaccuracies in the water boiler treatment or whether it arises from insufficient data for cylinders of larger diameter.

In summary, this report has attempted to derive a simple expression for the relationship between height at criticality and diameter in certain cylindrical geometries and to obtain, from this relationship, the limiting dimensions of infinite cylinders and slabs for comparison with results obtained by purely theoretical methods.

The authors are indebted to Dr. A. D. Callihan of the K-25 Laboratories for many helpful suggestions during the preparation of this report. The graphs appearing in this report were prepared by the Y-12 Engineering Department.

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APPENDIX

The empirical equation $(D-a)(H-b) = c$ was passed through a distribution of experimental points by the following curve fitting technique.[9]

Statement of the Theorem:

If two variables x and y are so related that the points represented by $x - x_k$, and $\frac{x - x_k}{y - y_k}$ lie on a straight line, the relation between the variable can be expressed by the equation $(x-a)(y-b) =$ constant.

If

$$(x-a)(y-b) = c \tag{1}$$

Let

$$x - x_k = X$$

$$y - y_k = Y$$

$$x = X + x_k$$

$$y = Y + y_k$$

(1) becomes

$$(X + x_k - a)(Y + y_k - b) = c \tag{2}$$

multiplying and collecting terms

$$XY + (x_k - a)Y + (y_k - b)X + (y_k - b)(x_k - a) = c \tag{3}$$

But $(y_k - b)(x_k - a) = c$

$$XY + (x_k - a)Y + (y_k - b)X = 0 \tag{4}$$

Dividing by $Y(y_k - b)$ (4) becomes

$$X/Y = - \frac{X}{y_k - b} - \frac{(x_k - a)}{(y_k - b)} \tag{5}$$

9. Running, Theodore R., "Empirical Formulas", John Wiley & Sons, New York, New York.

The slope of this linear function X/Y as a function of X is

$$\text{slope} = -\frac{1}{y_k - b} \tag{6}$$

$$\text{and the intercept} = -\frac{(x_k - a)}{(y_k - b)} \tag{7}$$

A sample calculation follows.

Sample Calculation:

1. To pass the empirical equation (D-a)(H-b) = c through a distribution of experimental points, critical height as a function of the cylinder diameter at a $H:U_{235}$ ratio of 31.6.

Since the experimental values of the diameter are given in inches it is convenient to leave them in inches until the final value of the infinite cylinder is obtained.

TABLE VI

D"	6.5	7.0	8.0	9.0	10.0
H cms	49.0	34.0	22.6	18.1	15.3
X"	0	.5	1.5	2.5	3.5
Y cms	0	-15.0	-26.4	-30.9	-33.7
X/Y	--	- .0333	- .0568	- 0809	- .1038

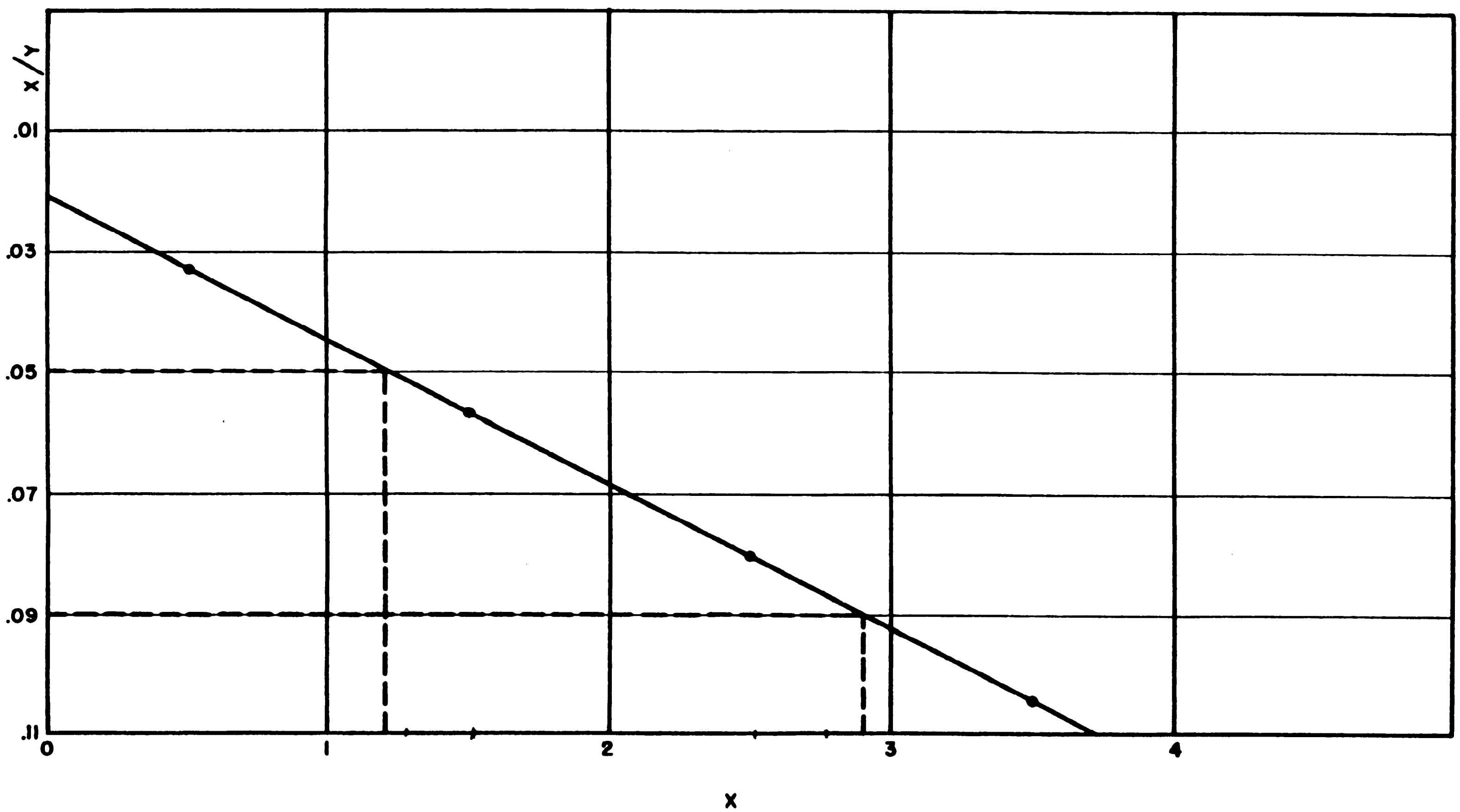
where X = (D - 6.5) and Y = H - 49.0

$$\text{Slope} = -\frac{1}{49.0 - b} = .0235$$ (obtained from Figure 8 and the preceding theoretical discussion)

b = 6.42 cms where b is the infinite slab thickness at a moderation of 31.6

$$\text{Intercept} = \frac{-6.5 - a}{49.0 - b} = -.021$$

Substituting for b, a then becomes equal to 5.61 inches of 14.2 cm where a is the diameter of the infinite cylinder.



$$SLOPE = \frac{\Delta X/Y}{\Delta X}$$

FIGURE 8

X/Y AS A FUNCTION OF X

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This equation then becomes

$$(D - 14.2)(H - 6.42) = c \tag{8}$$

The constant c is obtained by substituting the critical cylinder dimensions for a given $H:U_{235}$, equal to 31.6 in this case, into the above equation as follows.

From table VI, the 6.5 inch cylinder was critical when the height of solution was 49.0 cm

$$D = 6.5'' \text{ or } 16.51 \text{ cm}$$

$$H = 49.0 \text{ cm}$$

$$a = 14.2 \text{ cm}$$

$$b = 6.42 \text{ cm}$$

Equation (8) becomes

$$(16.5 - 14.2)(49.0 - 6.42) = c$$

$$c = 98.0 \text{ cm}^2$$

This procedure for determining c was applied to the data for 7", 8", 9" and 10" cylinder or the same $H:U_{235}$ ratio in Table I. The constants c were then averaged. The average value is 99.0. The equation for an $H:U_{235}$ = 31.6 then becomes

$$(D - 14.2)(H - 6.42) = 99$$

These are values reported in Table I in the main body of the report. The steps described here were repeated for various values of the $H:U_{235}$ ratio.

2. To calculate the minimum critical mass of a reactor at a given $H:U_{235}$

$H:U_{235}$ ratio = 50

$(D-a)(H-b) = 104$ (from Table II)

If $H = D$,

then $(D - 6.12)(D - 14.0) = 104$

from which

$D = 20.37$ cm or 8.02"

Critical volume = 6.64 l

Critical mass = .480 x 6.64 = 3.18 kilograms.

The calculations were repeated for other moderations and the resulting minimum critical masses are plotted against $H:U_{235}$.

TABLE VII

<u>Physical Constants Used in the Grueling Calculations</u>

η_{235}	=	2.50 neutrons/fission
$(\sigma_f)_{235}$	=	546 barns
$(\sigma_a)_{235}$	=	642 barns
σ_a/σ_f	=	1.18
$\nu_{235} = \eta/1.18$	=	2.12 neutrons/absorption
$(\sigma_a)_{238}$	=	2.56 barns
$(\sigma_a)_H$	=	0.32 barns
$(\sigma_a)_o$	=	.001 barns
$(\sigma_a)_F$	=	.01 barns
$(\sigma_s)U_{235}$	=	8.2 barns
$(\sigma_s)_H$	=	21 barns
p	=	.934 = isotopic purity

The subscripts f, a, and s refer to fission, absorption and scattering, respectively.